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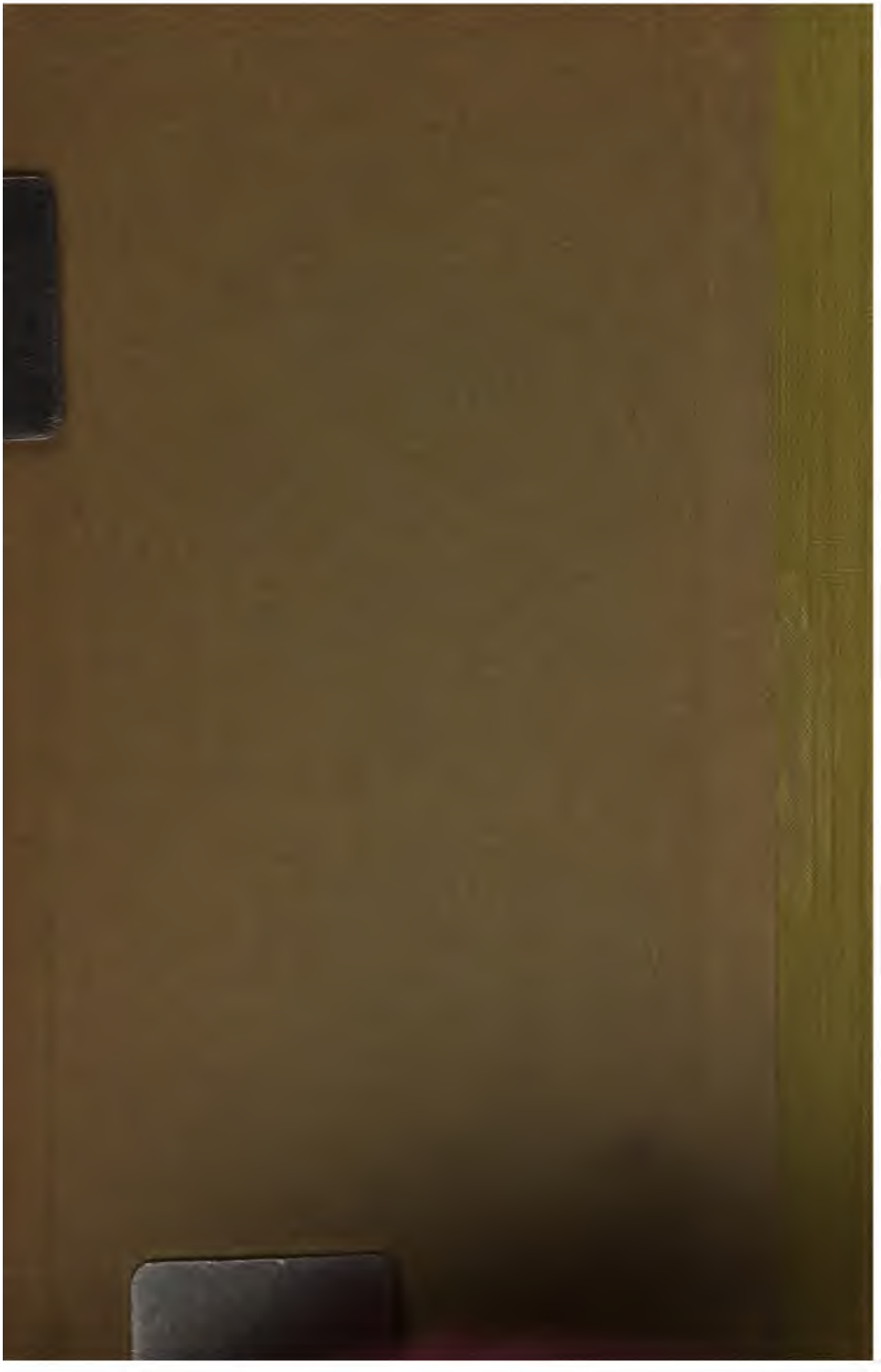

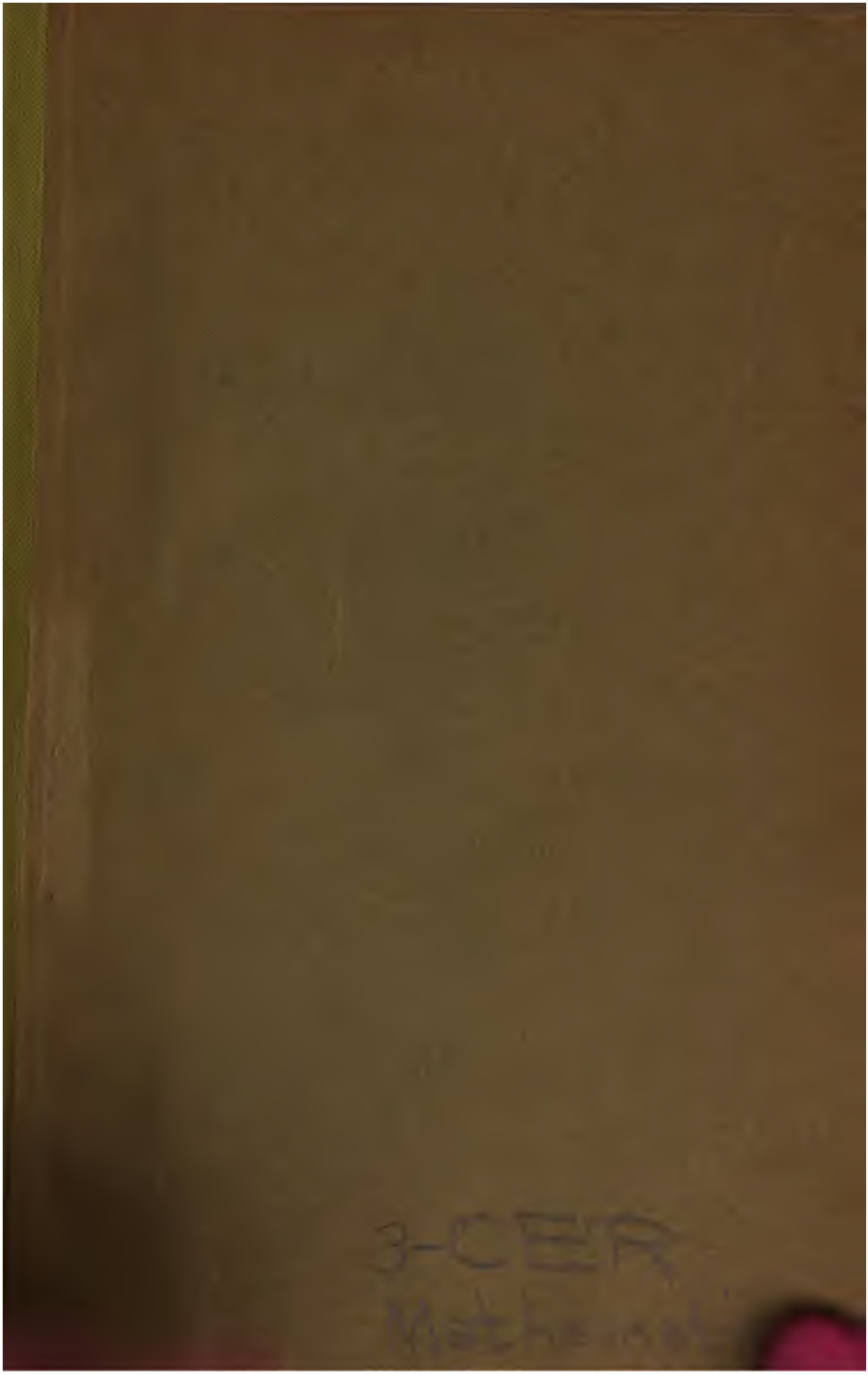
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WITH THEIR

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EDITED BY

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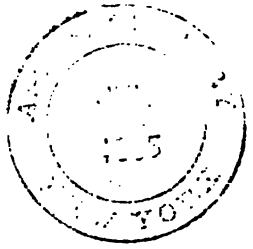

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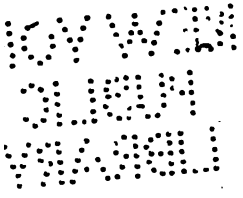

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Solved Questions.

1971. In a given tetrahedron, to find a set of rectangular coordinate axes, such that each axis shall pass through a pair of opposite edges 68

2100. 1. From one end O of a fixed diameter OD of a circle, two chords OA and OB are drawn such that the rectangle OA . OB = a constant (k^2) ; show that the envelope of the line AB is a circle with centre O and radius a third proportional to OD and k.

2. On a chord OA of a circle OAD, a point C is taken such that the sum or difference of the squares on OC and OA = a constant (k^2) ; show that the envelope of the perpendicular drawn to OC through C is a conic having O as centre 97

2267. If ∇ represent the determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ and if a', b', c', f', g', h', be the minors which respectively are the coefficients of a, b, c, f, g, h, in the expansion of ∇ : then show that the coordinates (x, y) of the foci of $(abcfgh)$ $(xy1)^2=0$, are given by

$$2c'x-2g' = \nabla^{\frac{1}{2}}\left\{(a-b+2hi)^{\frac{1}{2}}+(a-b-2hi)^{\frac{1}{2}}\right\},$$

and $$2c'yi-2f'i = \nabla^{\frac{1}{2}}\left\{(a-b+2hi)^{\frac{1}{2}}-(a-b-2hi)^{\frac{1}{2}}\right\},$$

where i represents $\sqrt{-1}$. Also show that $(c'x-g')$ $(c'y-f')$ $= \nabla h$ is the equation to an hyperbola concentric with the given conic and passing through its four foci 78

2384. 1. Construct a triangle of given species, having its vertices on three concentric circles.

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2. The triangle being equilateral, what relation exists between the radii when only one such triangle can be so constructed? 98

2348. In the system of conics touching two lines at fixed points, the focal cubic cuts a conic in points such that its normals at these points pass through the point of intersection of the two lines. 28

2364. Through each pair of the angles of a triangle circles are described so that the internal segments (in reference to the triangle) are similar. Find the locus of the intersection of the radical axes of the three circles. 29

2382. Given a quartic curve Q, and a point on it P; suppose that from p there are drawn the ten tangents to Q, omitting that at P, touching Q in ten points (t) and meeting it again in ten points (r). The t points, we know, lie on a cubic K; prove that the r points lie on a quartic U, with a point of osculation at p, touching K at p (counting for four points of intersection), and in four other points 67

2383. A and B are fixed points with regard to a conic, ACD a variable straight line passing through A and cutting the curve in C, D. The polar of A meets BC, BD in E, F; show that DE and CF meet in a fixed point G, and that ABG is a straight line 81

2407. Show (1) that the average distance of all the pairs of points that can be taken within a circle, in parts of the radius, is $\frac{128}{45\pi}$; and (2) that the same average for a sphere is $\frac{36}{35}$ 76

2413. Three planes at right angles to each other, and intersecting in a point A, touch an ellipsoid whose centre is B. Prove that the connector AB and the normals at the three points of contact are generators of a hyperboloid. 37

2414. 1. The number of ways in which n things, all different, can be allotted to r different persons, blank lots being admissible, is r^n.

2. If blank lots be *not* admissible, the number of ways is the coefficient of x^n in the expansion of $(\epsilon^x-1)^r \lfloor n$.

3. The number of ways in which n things, all alike, can be allotted to r different persons, blank lots being admissible, is the number of combinations of $n+r-1$ things taken $r-1$ at a time.

4. If blank lots be *not* admissible, the number of ways is the number of combinations of $n-1$ things taken $r-1$ at a time. 50

2425. A heavy cone stands inverted on its vertex, its axis making an angle α with the horizon; find the magnitude and direction of the least power which, when applied at the centre of its base, will just sustain the cone in that position..... 66

2426. If F_n denote $\dfrac{1}{a+a^3+\ldots+a^{2n-1}}\dfrac{a^2+a^4+\ldots+a^{2n}}{}$, prove that, in general,

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(1) ..... $F_n > \frac{n+1}{n} + \left(\sqrt{a} \quad \frac{1}{\sqrt{a}}\right)^2$,

(2) ..... $F_n > \frac{1}{2} \cdot \frac{n+1}{n}\left(a+\frac{1}{a}\right)$, (3) ..... $\frac{n}{n+1} F_n$

increases with n. 31

2428. Prove that

(1)... $\frac{\Gamma(mx)}{\Gamma(nx)}\ (x=0) = \frac{n}{m}$;

(2)... $\frac{\Gamma(n)}{\Gamma(2n)} - \frac{\Gamma(n+2)}{\Gamma(2n+2)} \cdot \frac{\pi^2}{1.2} + \frac{\Gamma(n+4)}{\Gamma(2n+4)} \cdot \frac{\pi^4}{1.2.3.4} - \&c. = 0.$ 43

2430. A cycloid revolving about its axis generates a solid which moves in the direction of the axis through a fluid with a given velocity; compare the resistances in the two cases when the vertex and the base are respectively foremost. 30

2453. The total number of partitions of n different things (*i.e.* the number of ways in which n different things can be distributed into 1, 2, 3, ... or n indifferent parcels) is the coefficient of x^n in the expansion of $\lfloor n \cdot e^{e^x - 1}$ 62

2461. If a curve of the nth degree be translated in one direction, the centre of mean distances of the points common to any two positions of the curve lies always on a straight line. 48

2465. If the tangent drawn to the ellipse $a^2y^2 + b^2x^2 = a^2b^2$, at one end of a chord whose pole is (h, k), be represented by $a^2yy' + b^2xx' = a^2b^2$, find the equation to the tangent at the other end of the polar 66

2474. 1. If $\left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 = 0$, show that, when n is any positive integer,

$$\left(\cos\theta \cdot \frac{d}{dx} + \sin\theta \cdot \frac{d}{dy}\right)^n = \left(\cos n\theta \cdot \frac{d}{dx} + \sin n\theta \cdot \frac{d}{dy}\right)\left(\frac{d}{dx}\right)^{n-1};$$

and hence

2. Prove the theorem (due to Rankine and Stokes) that at every multiple point in a plane curve which fulfils the condition $\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0$, the branches make equal angles with each other 99

2482. A quadrilateral circumscribes an ellipse, the eccentric angles of the points of contact being $\alpha, \beta, \gamma, \delta$. Prove that the condition that a pair of opposite vertices should lie on a confocal conic may be thus expressed:—

a^2 (cos A + cos B + cos C + cos D) (cos A + cos B − cos C − cos D) ×
(cos C + cos A − cos B − cos D) (cos B + cos C − cos A − cos D) =
b^2 (sin A + sin B + sin C + sin D) (sin A + sin B − sin C − sin D) ×
(sin C + sin A − sin B − sin D) (sin B + sin C − sin A − sin D;

where $A = \frac{1}{2}(-\alpha+\beta+\gamma+\delta)$, $B = \frac{1}{2}(\alpha-\beta+\gamma+\delta)$,
$C = \frac{1}{2}(\alpha+\beta-\gamma+\delta)$, $D = \frac{1}{2}(\alpha+\beta+\gamma-\delta)$ 65

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2485. In an acute-angled scalene triangle to inscribe the rectangle of minimum diagonal, having one of its sides coincident with a side of the triangle 109

2486. Find a point on an ellipse such that the normal produced to meet the curve again may cut off a maximum area. 56

2497. Show that if $a, b, c \ldots h$ denote

$$\frac{1}{1.2}\cdot\frac{d^2y}{dx^2},\ \frac{1}{1.2.3}\cdot\frac{d^3y}{dx^3},\ \frac{1}{1.2.3.4}\cdot\frac{d^4y}{dx^4},\ \ldots\ \frac{1}{1.2\ldots9}\cdot\frac{d^9y}{dx^9},$$

the general differential equation of

$a_1x^3+a_2x^2y+b_1xy^2+b_2y^3+a_3x^2+dxy+b_3y^2+c_1x+c_2y+c_3=0$ is

$$\begin{vmatrix} a, & b, & c, & a^2 & , & 0 & , & 0 \\ b, & c, & d, & 2ab & , & a^2 & , & 0 \\ c, & d, & e, & 2ac+b^2 & , & 2ab & , & a^3 \\ d, & e, & f, & 2ad+2bc & , & 2ac+b^2 & , & 3a^2b \\ e, & f, & g, & 2ae+2bd+c^2 & , & 2ad+2bc & , & 3a^2c+3ab^2 \\ f, & g, & h, & 2af+2be+2cd, & & 2ae+2bd+c^2, & & 3a^2d+6abc+b^3 \end{vmatrix} = 0 \quad 47$$

2499. ABCD is a quadrilateral; E, F, G, H the middle points of AB, BC, CD, DA; and $m_1, m_2, \ldots . m_8$ the middle points of AE, EB, BF, ... HA. Show that the sum of the squares of the 28 lines joining every two of the points $m_1, m_2, \ldots . m_8$ is

$$4(AC^2+BD^2)+AB^2+BC^2+CD^2+DA^2 \ldots\ldots 67$$

2503. Prove that the number of normals that can be drawn from a point in space of p dimensions to a (hyper)$^{p-3}$ surface (*i.e.*, what corresponds to a *surface* in space of three dimensions) of the nth degree is $\frac{n}{n-2}\left\{(n-1)^p-1\right\}$ 100

2506. Determine the two positions of the line whose chord of intersection with an assigned one of three given circles shall be cut harmonically by the other two 59

2511. If on an ordinary chessboard we compare any two rows or any two columns, squares opposite one another will be all of unlike or all of like colours according as the number of rows or columns intervening is even (zero included) or odd.

1. Show that it is possible to paint the squares in a manner such that, on comparing any two rows or columns, four of the squares opposite one another shall have like and the remaining four unlike colours.

2. Extend the construction to the case of a chessboard containing 2^i (instead of eight) squares in each direction..... 74

2514. Let a given volume of space, having a convex boundary *of any form whatever*, be referred to its centre of gravity and the principal axes of rotation, and let M denote the total volume, and h, k, l what may be called compressed radii of gyration round the axes, so that

$$M.h^2=\iiint x^2\,dx\,dy\,dz,\ M.k^2=\iiint y^2\,dx\,dy\,dz,\ M.l^2=\iiint z^2\,dx\,dy\,dz;$$

then

1. If *four* points be taken at random within the given vo-

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lume, the average square of the volume of the tetrahedron which connects them is $\frac{2}{3}h^2k^2l^2$.

2. If *six* points be taken at random within the given volume as the corners of a hexahedron, the probability that two of the points shall be re-entrant is $\frac{10h^2k^2l^2}{M^2}$. 24

2518. 1. Let P be a point on a given plane curve (A), O a fixed point, and Q the vertex of an equilateral hyperbola, whose centre is O, and which touches the given curve at P. Prove that the tangent at Q to the curve which is the locus of Q makes with OQ an angle equal to that made with OP by the tangent to (A) at P.

2. The curve (B), which is the inverse of the locus of Q with respect to the origin O, will be reciprocal to the given curve (A); that is to say, if (B) be regarded as the given curve, the original curve (A) will be derived from it in precisely the same manner as it has been derived from (A).

[A conic section referred to its centre, and an ellipse of Cassini, are reciprocal curves of this kind.] 38

2519. Given the six distances between four points in a plane; find the condition that one of them should be within the triangle formed by the other three. 19

2521. Two lines in space intersect a plain conic; show that the ruled surface determined by the three as directors is a quadric. 63

2523. If a, b, c be three closed areas situated in such a manner that no straight line can be drawn meeting more than two of them, the average area of all triangles formed by joining three points taken at random in A, B, C respectively, is equal to the area of the triangle formed by joining the centres of gravity of A, B, C.

2. If A, B, C, D be four closed volumes such that no plane can meet more than three of them, the average volume of the tetrahedron formed by joining four points taken at random in A, B, C, D respectively is equal to the volume of the tetrahedron formed by joining the centres of gravity of A, B, C, D respectively. 46

2532. If within a given enclosed area three points be taken at random as the apices of a triangle, the average of the square of the area of all such triangles will be reduced to one-third the value if one of the points be fixed at the centre. Also, if within a given volume of space four points be taken at random as the apices of a tetrahedron, the average of the square of the volume of all such tetrahedra will be reduced to one-fourth the value if one point be fixed at the centre. Show that this theorem is true when the enclosed area or volume of space is *of any form whatever*. 24

2535. The ends of any diameter of an ellipse are joined with one of the foci; prove that the angles made by the joining lines with the major axis are the eccentric angles corresponding to the ends of a focal chord. 32

2548. Given $\cos^2 x \cos^2 a = \sin^2 a \cos^2 a - 4 \cos x \sin (x-a) \sin^3 a$; it is required to find the angle x. 30

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2551. Given the partial differential equation

$$\frac{d^2z}{dy^2} - a^2\frac{d^2z}{dx^2} + \beta\frac{dz}{dy} + \gamma z = 0 \quad \text{.....(I)},$$

wherein α, β, and γ are functions of x only. Show that when the condition

$$\gamma - \frac{1}{4}\left(\frac{d\alpha}{dx} + \beta\right)^2 - \frac{\alpha^2}{2}\frac{d}{dx}\left\{\frac{1}{\alpha}\left(\frac{d\alpha}{dx} + \beta\right)\right\} = 0 \quad \text{.....(II)}$$

is satisfied, then in general $z = \sqrt{\alpha}\,\epsilon^{\int \frac{\beta}{2\alpha}dx}\,\phi\left(y + \int\frac{dx}{\alpha}\right)$,

where ϕ is an arbitrary function. Show also that, when β is constant and (II) satisfied, then the integral of (I) is

$$x = \sqrt{\alpha}\,\epsilon^{\int \frac{\beta}{2\alpha}dx}\,\phi\left(y + \int\frac{dx}{\alpha}\right) + k\epsilon^{-\int \frac{\beta}{2\alpha}dx}\,\psi\left(y - \int\frac{dx}{\alpha}\right),$$

where ψ is another arbitrary function, and the arbitrary multiplier k is only inserted for convenience 81

2556. Four solids, *having any forms whatever*, being given, show how to determine sets of six points fixed in each, such that if the four solids be placed in any positions in space and four points be taken arbitrarily, one in each solid, as the apices of a tetrahedron, the average square of volume in respect of all such tetrahedra shall always be equal to that of the 1296 tetrahedra formed by connecting all the groups obtainable from the four invariable sets of six points. 24

2561. 1. A point is taken at random on each side of an equilateral triangle; show that the chance that these points form an acute-angled triangle is $1 - \frac{9}{2}\log\left(\frac{3}{64}\epsilon\right)$.

2. Find also the chance that a circle passing through the three points cuts each side of the triangle (not produced) in a second point. 21

2568. Express the roots of the equation

$$a^6x^6 + 18a^4(bd - c^2)x^4 + 81a^2(bd - c^2)^2x^2 + 27d^2(a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd) = 0$$

in terms of α, β, γ, the roots of the equation

$$ax^3 + 3bx^2 + 3cx + d = 0. \text{ ..... } 44$$

2571. The equation whose roots are the ratios of the roots of $ax^3 + 3bx^2 + 3cx + d = 0$ is

$$a^2d^2x^6 + (3a^2d^2 - 9abcd)x^5 + (6a^2d^2 - 45abcd + 27ac^3 + 27b^3d)x^4 + (7a^2d^2 - 54abcd + 54ac^3 - 81b^2c^2 + 54b^3d)x^3 + (6a^2d^2 - 45abcd + 27ac^3 + 27b^3d)x^2 + (3a^2d^2 - 9abcd)x + a^2d^2 = 0.$$

Prove this, and show that its discriminant is

$$a^4d^4(ad - 9bc)^2(b^3d - ac^3)^4(a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd)^3;$$

and account *à priori* for the circumstance that, while in the equation b and c rise to the third power, a and d only reach the second. 45

2577. Show that the average area of all the ellipses that can be drawn touching a semicircle and its diameter symmetrically, in parts of the area of the ellipse, is $\frac{\pi}{16}(2 + \sqrt{2})$, or ·45523..., if (1) all possible positions of the centre of the ellipse be sup-

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posed equally probable, and $(2-\sqrt{2})$, or ·58579..., if (2) all distances of the chord of contact from the centre of the semi-circle be supposed equally probable 84

2579. If a line and circle are given, and Q is a point on the tangent to the circle at P so that PQ is bisected by the line, the locus of Q is a quartic with three double points, each branch having an inflexion as it passes through a double point. If the line touches the circle, the quartic becomes the line and a Cissoid of Diocles.

2. Reciprocally, find the envelope of the chord of intersection of a central conic with its circles of curvature. What is it in the parabola? 91

2584. AQM is a semicircle, QC the vertical radius, AB a chord cutting QC in L so that $QC = n \cdot CL$; find the ratio between the triangles BCL and ACL. 32

2585. Four points are taken at random on the sides of a quadrilateral; what is the chance of the lines *joining* them intersecting? 38

2587. Writing $F(z)$ for $a_0 + a_1 z + \frac{a_2 z^2}{1.2} + \frac{a_3 z^3}{1.2.3}$ + &c., show that the coefficient of $x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}$ in $F(x_1 - x_2) \cdot F(x_2 - x_3) \ldots$ $\ldots F(x_m - x_1)$ is an invariant, or the leading term of a covariant relative to the function formed by taking the terms of $F(z)$ involved in the coefficient and substituting binomial for exponential coefficients 42

2591. Let AA_1, BB_1, CC_1 be the bisectors of the triangle ABC, intersecting in G_1; let AA_2, BB_2, CC_2 be the perpendiculars from the angular points on the opposite sides, intersecting in G_2; and let B_1C_1, B_2C_2 intersect in P; C_1A_1, C_2A_2 in Q; and A_1B_1, A_2B_2 in R. Then AP, BQ, CR will all be perpendicular to G_1G_2; and the triangle PQR will "circumscribe" the triangle ABC. Let A_i, B_i, C_i be the feet of the interior bisectors of the angles A, B, C, and A_e, B_e, C_e the feet of the exterior bisectors: then the six straight lines PA_i, QB_i, RC_i, PA_e, QB_e, RC_e pass three and three through four points *which are the points of contact of the nine-point circle with the inscribed and escribed circles*... 51

2604. Prove that, in the third system of pulleys, (these being supposed n in number, of equal weight w, and equal diameter $2a$,) the distance of the centre of the parallel tensions of the cords from the first cord is equal to

$$\frac{4\left\{(n-1)\,2^{n-1}+1\right\}W + n\left\{(n-3)\,2^n + n + 3\right\}w}{2\,(2^n-1)\,W}\,a. \quad \ldots\ldots \quad 55$$

2605. A horizontal beam AB, supported but not fixed at both ends A and B, is traversed from end to end by a moving load W distributed uniformly over a segment of it of constant length PQ. Show that the greatest bending moment M at any point X of the beam corresponds to the position of the load in its

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passage over the point for which PX : QX = AX : BX, and has for value

$$M = \frac{W}{AB^2} \cdot AX \cdot BX \cdot (AB - \tfrac{1}{2}PQ) \quad \ldots\ldots\ldots\ldots \quad 60$$

2607. In three straight lines given in position, to determine the positions of the vertices of a triangle of given or minimum magnitude, and of given species. 63

2609. Given three conics passing through the same four points; and on the first a point A, on the second a point B, and on the third a point C. It is required to find, on the first a point A′, on the second a point B′, and on the third a point C′, such that the intersections of the lines

A′B′ and AC, A′C′ and AB, lie on the first conic;
B′C′ and BA, B′A′ and BC, lie on the second conic;
C′A′ and CB, C′B′ and CA, lie on the third conic. 17

2612. If three points P, Q, R be taken at random, one on each of the sides BC, CA, AB, of the triangle ABC, the chance that the area of the triangle PQR shall be greater than $(n+\frac{1}{4})$ of the area of the triangle ABC (n being positive and $<\frac{3}{4}$) is

$$\frac{3-4n}{4} - \frac{12n+1}{4} \log \frac{4}{4n+1} + 8n^{\frac{3}{2}} \left(\tan^{-1} \frac{1}{2n^{\frac{1}{2}}} - \frac{\pi}{6} \right) \ldots\ldots \quad 109$$

2614. Three points are taken at random, one on each of three faces of a tetrahedron, show that the chance that the plane passing through them cuts the fourth face is $\frac{3}{4}$..... 54

2620. 1. A rigid body is rotating round a fixed axis; suddenly this axis becomes free, and a parallel axis becomes fixed elsewhere in the body. Find the new initial angular velocity, and the locus of the new axis if the angular velocity is unchanged.

2. A right prism, whose base is a regular polygon of n sides, falls (without sliding) down a rough inclined plane, the contact being supposed to take place only at the corners; find its angular velocity just before the mth impact.

3. Hence deduce the law of motion of a cylinder or sphere down a rough inclined plane..... 22

2624. If ABC, A″B″C″ be two triangles, and T any point such that the area of the triangle $(\alpha\beta\gamma)$ formed by the intersections (α, β, γ) of TA″, TB″, TC″ respectively with BC, CA, AB, is constant; then the locus of T is a cubic having only one arbitrary parameter..... 79

2626. To draw through a given point within a circle a chord which shall be divided at the point in extreme and mean ratio..... 23

2627. How many different signals can be made with a set of n different flags, using r of them at a time, and arranging them on s masts, where $n > r > s$?..... 59

2630. Show that the degree of the locus of the foci of a system of conics, subject to any four conditions, is three times as great as that of the locus of their centres 63

2631. A conic F has double contact with one of a set of conics having double contact with one another. Show that the

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chords of intersection of F with each of the set will intersect on the common chord of contact of the whole set 78

2639. The direction angles of three lines through the origin are $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$; required the equation of the cone of revolution of which they are sides 62

2641. Find the value of the double integral

$$\iint \frac{dx\,dy}{\left\{(x^2+y^2+a^2-b^2)^2-4(a^2-b^2)x^2\right\}^{\frac{1}{2}}},$$

subject to the condition $\frac{x^2}{a^2}+\frac{y^2}{b^2}<1$. 20

2646. A cube being cut by any plane, what is the chance that the section is a hexagon? 36

2647. In the equation $x^3-qx+r=0$ it is known that q and r both lie between $+1$ and -1; assuming all values between these limits to be equally probable, prove that the chance of all the roots of the equation being real is $\frac{3}{4\pi}\sqrt{3}$. 43

2661. Given five conics, it is of course possible in an infinity of ways to determine five constants a, b, c, &c., so that

$$aU_1+bU_2+cU_3+dU_4+eU_5$$

may be either a perfect square L^2 or the product of two factors MN. Prove that the lines L all touch a conic, and that the lines M, N are conjugate with regard to that conic; from which it follows that, if M be given, N passes through a fixed point. 19

2662. Prove that

$$\frac{\sin n\alpha \sin(\beta-\gamma)+\sin n\beta \sin(\gamma-\alpha)+\sin n\gamma \sin(\alpha-\beta)}{\sin(\gamma-\beta)+\sin(\beta-\alpha)+\sin(\alpha-\gamma)} = 2\Sigma(p\alpha+q\beta+r\gamma);$$

p, q, r being positive integers whose sum is n, no two being zero, and the coefficient when one is zero being 1 instead of 2. 41

2669. If the diagonals d, d' of any plane quadrilateral be divided at their point of intersection in the ratios $p:p'$ and $q:q'$ respectively, and θ be the angle between them, then (1) the square of the third diagonal will be equal to

$$\frac{p^2p'^2(q^2-q'^2)^2d^2+q^2q'^2(p^2-p'^2)^2d'^2+2pp'qq'(p^2-p'^2)(q^2-q')dd'\cos\theta}{(pq-p'q')^2(pq'-p'q)^2};$$

and (2) the line which divides these diagonals in the ratios $m:m'$ and $n:n'$ respectively will divide the third diagonal in the ratio $\frac{mnp'q'-m'n'pq}{p'q'-pq} : \frac{m'npq'-mn'p'q}{pq'-p'q}$ 39

2675. If two triangles lie in perspective, prove that a cubic curve may be drawn through the six angles, such that for each triangle the tangents at its angles shall all three meet together on the curve. 35

2676. Let the sides of the triangle of reference subtend at a point $(\alpha', \beta', \gamma')$ angles $(\theta, \theta', \theta'')$. Then

$$\sin\theta\left(\frac{\alpha}{\alpha'}\right)^{\frac{1}{2}}+\sin\theta'\left(\frac{\beta}{\beta'}\right)^{\frac{1}{2}}+\sin\theta''\left(\frac{\gamma}{\gamma'}\right)^{\frac{1}{2}}=0$$

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is a conic touching the three lines and having $(\alpha', \beta', \gamma')$ for focus.

Also $\tan \frac{1}{2}\theta \left(\frac{\alpha'}{\alpha}\right) + \tan \frac{1}{2}\theta' \left(\frac{\beta'}{\beta}\right) + \tan \frac{1}{2}\theta'' \left(\frac{\gamma'}{\gamma}\right) = 0$

is a conic circumscribing the triangle and having $(\alpha', \beta', \gamma')$ for focus.

If we denote by λ, μ, ν the perpendiculars drawn from the vertices of the triangle of reference on any tangent, and ρ, ρ', ρ'' the distances of $(\alpha', \beta', \gamma')$ from the vertices; then

$$\sin \theta \left(\frac{\rho}{\lambda}\right) + \sin \theta' \left(\frac{\rho'}{\mu}\right) + \sin \theta'' \left(\frac{\rho''}{\nu}\right) = 0$$

is the tangential equation of the first of these conics, and

$$\sin \tfrac{1}{2}\theta \left(\frac{\lambda}{\rho}\right)^{\frac{1}{2}} + \sin \tfrac{1}{2}\theta' \left(\frac{\mu}{\rho'}\right)^{\frac{1}{2}} + \sin \tfrac{1}{2}\theta'' \left(\frac{\nu}{\rho''}\right)^{\frac{1}{2}} = 0$$

is the tangential equation of the second. 34

2680. If A and B be two points taken at random within a closed convex figure, and the line AB be produced to meet the boundary in P and Q, prove that the chance that the three segments PA, AB, BQ are such as would form a triangle is $\frac{1}{4}$. 90

2682. If circles be described on a pair of opposite sides of a square; prove geometrically that

1. The polars of any point on either diagonal with respect to the two circles meet on the other diagonal.

2. The four tangents from the point form a harmonic pencil 101

2687. If normals to circumscribing conics at the vertices of a triangle meet in a point, and if also normals to inscribed conics at the points of contact meet in a point; show that the locus of this point is the same in both cases 58

2690. If four points on a plane be the extremities of the two straight lines a, a' or of the pair b, b', or of the pair c, c'; then, a, b, c being the sides of a triangle, prove that

$$(a^2a'^2 + b^2b'^2 + c^2c'^2)\ (a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2)$$
$$= 2\,(a^4a'^2 + a'^4a^2 + b^4b'^2 + b'^4b^2 + c^4c'^2 + c'^4c^2) + a^2b^2c^2 + a^2b'^2c'^2 + a'^2b^2c'^2 + a'^2b'^2c^2. \quad 57$$

2694. A rope without weight passes over a smooth pulley, having a monkey at one end and a weight equal to the monkey at the other. The monkey suddenly runs up the rope to a height h with a uniform velocity; what is the resulting motion of the weight? 53

2695. If a given finite straight line of length a be divided at random in two points, the chance that the product of the three parts shall exceed $\frac{1}{108}a^3$ is

$$\frac{4}{9}\int_{\frac{4}{9}\pi}^{\frac{8}{9}\pi} \sin \tfrac{1}{2}\theta\,(1 + 2\cos 3\theta)^{\frac{1}{2}} d\theta \quad \text{..... } 58$$

2696. Let $G = 1 + \theta + \theta^2 + \theta^3$ be any group of powers of a substitution made with the elements 1234. Let H_1 be any asymmetric function of $x_1x_2x_3x_4$. And let $\theta; H_1$ or $G; H_1$ be the symbol

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of the result of operation on H_1 by θ or by the whole group G. Thus $G ; H_1$ is an algebraic sum of four values of H_1.

1. There is a substitution ϕ such that

$$\left\{(G ; H_1^i)^j + \phi ; (G ; H_1^i)^j\right\}^k$$

is a function which has not more than three values, all given with G and H_1, whatever be the integer exponents i, j, k.

2. Hereby the algebraic solution of the general quintic equation is attainable at the cost of few words 70

2697. Let I be a point of inflexion on a cubic. From I draw tangents meeting the cubic in P, Q, R; and from P draw tangents meeting it in A, B, C, D. Prove that the three opposite pairs of sides of the tetragon ABCD will respectively intersect in I, Q, R. 53

2702. The locus, in areal coordinates, of the centre of an ellipse, of constant area (πn^2), circumscribing a triangle, is

$$(x+y+z)\,(x+y-z)\,(x-y+z)\,(-x+y+z) = \frac{4x^2y^2z^2}{n^4} \quad \ldots\ldots 77$$

2704. A central conic ($a^2y^2 + b^2x^2 = a^2b^2$) is turned about its centre in its own plane through a right angle; prove that the locus of the intersection of any tangent to the given conic with the same line in its new position is one of the two quartics

$$(x^2+y^2)^2 = a^2(x \mp y)^2 \pm b^2(x \pm y)^2 \quad \ldots\ldots 111$$

2718. Find *in plano* the locus of a point P, such that from it two given points A, B, and two given points, C, D subtend equal angles 106

2721. Given any convex area Ω, let C be the length of any chord drawn across the area, and let the position of C be defined by (p, θ), the perpendicular on it from a fixed pole inside Ω and the inclination of p to a fixed axis; then $\iint C^3\,dp\,d\theta = 3\Omega^2$, the integration extending to all possible positions of C 89

2732. An epi- or hypo-cycloid is pushed through a very short fixed tube, so as to remain in one plane, show that the locus of its centre is an ellipse 96

2733. Show that the average area of a series of triangles, equably distributed over the surface of an ellipse so as to have each one side parallel to a given line, in parts of the area of the ellipse, is $\frac{256}{525\pi^2}$ 92

2737. Find *in solido* the locus of a point P, such that from it two given points, A, C, and two given points B, D, subtend equal angles. 108

2738. The equation representing the transmission of sound parallel to the axis of a tube, when formed without reference to any previously assumed law of pressure, is $\frac{d^2y}{dt^2} + \frac{1}{D} \cdot \frac{dp}{dx} = 0$, where x is the ordinate of the point of rest of a particle, y the same ordinate at the time t, p the pressure at the same time, and D the density of equilibrium. Show that the equation is

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satisfied by the values of the pressure, velocity, and density derived from the three following equations, viz.:

$$p = -\frac{a^2}{\rho} + \chi\left(v + \frac{a}{\rho}\right), \quad v + \frac{a}{\rho} = \phi\left(x + \frac{a - \chi'}{D}t\right),$$

$$\frac{1}{\rho} - f\left(v + \frac{a}{\rho}\right) = \psi\left(x - \frac{a}{D}t\right),$$

where a is an arbitrary constant; χ, ϕ, and ψ are arbitrary functions; and f is a function whose form is given by the equation

$$f(z) = \int \frac{dz}{2a - \chi'(z)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad 86$$

2743. Show that if p be a prime number and m and n any positive integers, the highest power of p contained in $\frac{\Pi(m+n)}{\Pi(m)\ \Pi(n)}$ may be obtained by expressing $m+n$ and either m or n in the scale of p; the number of times that it would be necessary to borrow in subtracting the latter number from the former being the index of the power of p required 88

2748. If a circular cubic with a double point O be cut by a circle in four points A, B, C, D; and if OA, OB, OC, OD cut the circle again in E, F, G, H; show that any pair of straight lines joining these four points will be equally inclined to the bisectors of the angles between the tangents at O 105

2756. Show that an infinite number of triangles can be described such that each has the same circumscribing, nine-point, and self-conjugate circles as a given triangle 98

CORRIGENDA.

Vol. VIII.

Page 82, lines 27 and 28, *for* as to sign *read* as to even and odd.

Vol. IX.

" 65, line 2 from bottom, *for* in solution *read* in this solution.
" 66, " 2, *for* xy *read* xy.
" 80, " 3 from bottom, *for* $BQ = x$ *read* $BQ = x_1$.
" 101, " 12, *insert* $=$ *before* $10 - \pi^2$.
" 107, " 18, *for* $\int z\,dx$ *read* $\int z\,dx$.

Vol. X.

" 17, " 10, *for* AB′ *read* A′B′.
" 31, " 10 from bottom, *insert* $-$ *after* $\sqrt{a}$.
" 42, " 12, *for* $a_1\frac{z^2}{1.2}$ *read* $\frac{a_2^2 z^2}{1.2}$.
" 64, " 30, *for* CA at K *read* BA at K.
" 72, " 16, *for* 2143 *read* 2134.
" 85, " 15, *for* APP′E *read* FPP′E.
" 92, last line, *insert* is *before* $\frac{256}{525\pi^2}$.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

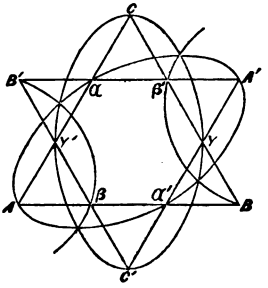
2609. (Proposed by Professor CAYLEY.)—Given three conics passing through the same four points; and on the first a point A, on the second a point B, and on the third a point C. It is required to find, on the first a point A, on the second a point B′, and on the third a point C′, such that the intersections of the lines

A B′ and AC, A′C′ and AB, lie on the first conic;
B′C′ and BA, B′A′ and BC, lie on the second conic;
C′A′ and CB, C′B′ and CA, lie on the third conic.

Solution by the PROPOSER.

Let the six intersections in question be called α, α'; β, β'; γ, γ', respectively; then BC intersects second conic in β', third in γ, CA intersects third conic in γ', first in α, AB intersects first conic in α', second in β; and we have—

A′ the intersection of $\alpha\beta'$, $\gamma\alpha$,
B′ the intersection of $\beta\gamma'$, $\alpha\beta'$,
C′ the intersection of $\gamma\alpha'$, $\beta\gamma'$;



and it has to be shown that the points A′, B′, C′ so determined lie—A′ on the first conic, B′ on the second conic, C′ on the third conic.

Taking $x = 0$, $y = 0$, $z = 0$ for the equations of the sides of the triangle ABC, the equations of the three conics may be taken to be $U = 0$, $V = 0$, $W = 0$, where the functions U, V, W are such that identically $U + V + W = 0$; and then observing that the conics pass through the points $(y = 0, z = 0)$, $(z = 0, x = 0)$, $(x = 0, y = 0)$, respectively, we see that the equations may be taken to be

$$(\ 0, -b, \quad c, f_1, g_1, h_1)(x, y, z)^2 = 0,$$
$$(\ a, \quad 0, -c, f_2, g_2, h_2)(x, y, z)^2 = 0,$$
$$-a, \quad b, \quad 0, f_3, g_3, h_3)(x, y, z)^2 = 0,$$

where $f_1+f_2+f_3 = 0,\ g_1+g_2\times g_3 = 0,\ h_1+h_2+h_3 = 0.$

The coordinates of the points $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are at once found to be

$$\alpha, (\ c\ ,\ 0\ ,\ -2g_1);\quad \alpha' (b\ ,\ 2h_1\ ,\ 0)$$
$$\beta, (-2h_2,\ a\ ,\ 0);\quad \beta' (0\ ,\ c\ ,\ 2f_2)$$
$$\gamma, (\ 0\ ,\ -2f_3,\ b);\quad \gamma' (2g_3,\ 0\ ,\ a).$$

And hence the equations of $\beta\gamma'$, $\gamma\alpha'$, $\alpha\beta'$ are

$$\beta\gamma';\quad ax\ +2h_2y-2g_3z = 0,$$
$$\gamma\alpha\ ;-2h_1x+\ by+2f_3z = 0,$$
$$\alpha\beta';\quad 2g_1x-2f_1y+\ cz = 0.$$

Hence for the point A′, which is the intersection of $\gamma\alpha'$, $\alpha\beta'$, coordinates are

$$bc+4f_2f_3,\quad 4f_3g_1+2ch_1,\quad 4h_1f_2-2bg_1;$$

and A′ will be on the first conic if only

$$(0, -b, c, f_1, g_1, h_1)(bc+4f_2f_3,\ 4f_3g_1+2ch_1,\ 4h_1f_2-2bg_1)^2 = 0$$

viz., this equation is

$$\begin{aligned}
&-\ b\ (\ 16f_3^2g_1^2\ \ +16f_3g_1h_1c+4h_1^2c^2)\\
&+\ c\ (\ 16h_1^2f_2^2\ \ -16f_2g_1h_1b+4g_1^2b^2)\\
&+2f_1\ (\ 16c_1h_1f_2f_3-8g_1^2f_3b\ \ +8h_1^2f_2c-4g_1h_1bc)\\
&+2g_1\ (+16h_1f_2^2f_3\ \ -8g_1f_2f_3b+4h_1f_2bc-2g_1b\,c)\\
&+2h_1\ (+16g_1f_2f_3^2\ \ +8h_1f_2f_3c+4g_1f_3bc+2h_1bc^2) = 0,
\end{aligned}$$

viz, this is easily found to be

$$8\,(2g_1f_3+ch_1)\,(2h_1f_2-bg_1)\,(f_1+f_2+f_3) = 0,$$

which is satisfied in virtue of $f_1+f_2+f_3 = 0$; that is, A′ is on the first conic; and similarly, in virtue of $g_1+g_2+g_3 = 0$, B′ is on the second conic; and in virtue of $h_1+h_2+h_3 = 0$, C′ is on the second conic. But the same thing appears at once by the remark that the equations of the three conics are

$$\begin{aligned}
&-y\,(-2h_1x+\ by+2f_3z)+z\,(\ 2g_1x-2f_2y+\ cz) = 0,\\
&-z\,(\ 2g_1x-2f_2y+\ cz)+x\,(\ \ ax+2h_2y-2g_3z) = 0,\\
&-x\,(\ \ ax+2h_2y-2g_3z)+y\,(-2h_1x+\ by+2f_3z) = 0.
\end{aligned}$$

It may be added that, taking (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), $(x_4, y_4\ z_4)$ as the coordinates of the four points of intersection of the three conics, the first conic is given by means of these four points and the fifth point $(y = 0,\ z = 0)$; and similarly for the other two conics; whence, denoting the determinants formed with any four columns out of the matrix

$$\left\|\begin{array}{cccccc}
x_1^2, & y_1^2, & z_1^2, & y_1z_1, & z_1x_1, & x_1y_1\\
x_2^2, & y_2^2, & z_2^2, & y_2z_2, & z_2x_2, & x_2y_2\\
x_3^2, & y_3^2, & z_3^2, & y_3z_3, & z_3x_3, & x_3y_3\\
x_4^2, & y_4^2, & z_4^2, & y_4z_4, & z_4x_4, & x_4y_4
\end{array}\right\|$$

by 1234, 1235, &c., we easily find the equations of the three conics, viz., these may be written

$$\begin{array}{rccccccc} & x^2 & , y^2 & , z^2 & , yz & , zx & , xy \\ 1456 (& 0 & , +3456, & -2456, & +2356, & +2364, & +2345) = 0, \\ 2356 (& -3456, & 0 & , +1456, & +3156, & +3164, & +3145) = 0, \\ 3456 (& 2456, & -1456, & 0 & , +1256, & +1264, & +1245) = 0, \end{array}$$

the exterior factors 1456, 2356, 3456 being introduced in order to bring the equations into the above mentioned form, wherein the sum of the quadric functions is = 0.

2661. (Proposed by Dr. SALMON, F.R.S.)—Given five conics, it is of course possible in an infinity of ways to determine five constants a, b, c, &c., so that

$$aU_1 + bU_2 + cU_3 + dU_4 + eU_5$$

may be either a perfect square L^2 or the product of two factors MN. Prove that the lines L all touch a conic, and that the lines M, N are conjugate with regard to that conic; from which it follows that, if M be given, N passes through a fixed point.

Solution by the PROPOSER.

Given a pair of conics U and V, it is known that there is an invariant Θ of the system, of the first degree in the coefficients of U and of the second in those of V; and that if the coefficients of U be a, b, &c., and if the reciprocal coefficients of V be A, B, &c., Θ may be written

$$Aa + Bb + Cc + 2Ff + 2Gg + 2Hh.$$

If we are given five conics U, we have conditions enough to determine linearly A, B, &c., so that Θ may vanish for V and each of the five conics; and since this invariant contains the U coefficients in the first degree, a single conic V can be thus determined whose invariant Θ will vanish for every conic of the system

$$l_1V_1 + l_2U_2 + l_3U_3 + l_4V_4 + l_5U_5.$$

If this last be a perfect square L^2, the vanishing of Θ is the condition that L should touch V; or if it break up into factors MN, the vanishing of Θ is the condition that M, N should be conjugate with respect to V.

2519. (Proposed by C. W. MERRIFIELD, F.R.S.)—Given the six distances between four points in a plane; find the condition that one of them should be within the triangle formed by the other three.

Solution by the PROPOSER.

Let the triangles formed by the points taken three and three be A, B, C, D. Then, evidently, the criterion is, that the lower sign should be taken in the necessary condition $(A + B)^2 = (C \pm D)^2$, that is to say, that

$$(A^2 + B^2 + C^2 + D^2)^2 - 2(A^4 + B^4 + C^4 + D^4)$$

should be negative.

Next, let the sides (opposite) be lp, mq, nr, so that the four triangles may consist of the sets lmn, lqr, mpr, npq.

Then if we substitute $\frac{1}{16}\sigma\ (2l^2m^2+2m^2n^2+2n^2l^2-n^4-m^4-l^4)$ for A^2, &c., we get, as the condition that one point should lie within the triangle formed by the three others, that the following quantity should be negative:—

$$\left.\begin{aligned}
&2\Sigma\,(l^2m^2n^2\rho^2)+(l^4p^4+m^4q^4+n^4r^4)\\
&+l^4\left\{m^2q^2+n^2r^2-p^2\,(m^2+q^2+n^2+r^2)-m^2n^2-q^2r^2\right\}\\
&+p^4\left\{m^2q^2+n^2r^2-l^2\,(m^2+q^2+n^2+r^2)-m^2r^2-n^2q^2\right\}\\
&+m^4\left\{l^2p^2+n^2r^2-q^2\,(l^2+p^2+n^2+r^2)-l^2n^2-p^2r^2\right\}\\
&+q^4\left\{l^2p^2+n^2r^2-m^2\,(l^2+p^2+n^2+r^2)-l^2r^2-n^2p^2\right\}\\
&+n^4\left\{l^2p^2+m^2q^2-r^2\,(l^2+p^2+m^2+q^2)-l^2m^2-p^2q^2\right\}\\
&+r^4\left\{l^2p^2+m^2q^2-n^2\,(l^2+p^2+m^2+q^2)-l^2q^2-m^2p^2\right\}
\end{aligned}\right\}\ \ldots\ldots(1).$$

The law of this is easily seen. Its form may be modified by means of the relation between the distances of four points in a plane, given by Carnot, viz.,

$$\left.\begin{aligned}
&l^2m^2n^2+l^2q^2r^2+m^2p^2r^2+n^2p^2q^2\\
&+l^2p^2\ (l^2+p^2-m^2-n^2-q^2-r^2)\\
&+m^2q^2\ m^2+q^2-l^2-n^2-p^2-r^2)\\
&+n^2r^2\ (n^2+r^2-l^2-m^2-p^2-q^2)=0
\end{aligned}\right\}\ \ldots\ldots\ldots(2),$$

which simply expresses that the tetrahedron degenerates into a plane.

The consideration that the form (1) must vanish whenever any one of the four triangles A, B, C, D vanishes leads to the inference that the form is reducible to a function of the continued product of sixteen factors,

$$(\pm\,l\pm m\pm n)\,(\pm\,l\pm q\pm r)\,(\pm\,m\pm p\pm r)\,(\pm\,n\pm p\pm q)\ \ldots(3),$$

that is to say, of one of the (hemihedral) square roots of the (holohedral) expression of thirty-two factors. In fact the equation

$$(A^2+B^2+C^2+D^2)^2-2\,(A^4+B^4+C^4+D^4)=\mp\,8ABCD\ \ldots\ldots(4),$$

which is a necessary consequence of $(A+B)^2=(C\pm D)^2$,

and in which the value of $16A^2$ may be written as

$$-(l+m+n)\,(l-m-n)\,(-l+m-n)\,(-l-m+n),$$

(and so for the others), shows that (1) may be reduced to the square root of (3) by the help of (2), if not without it. This remark may have some algebraical value. Meanwhile it is to be observed that it is the right-hand side only of equation (4) which yields the criterion.

2641. (Proposed by the Rev. W. Roberts, M.A.)—Find the value of the double integral $\iint \frac{dx\,dy}{\left\{(x^2+y^2+a^2-b^2)^2-4\,(a^2-b^2)\,x^2\right\}^{\frac{1}{2}}}$, subject to the condition $\frac{x^2}{a^2}+\frac{y^2}{b^2}=<1$.

Solution by M. W. CROFTON, F.R.S.

Using elliptic coordinates μ, ν [the semi-transverse axes of the ellipse and hyperbola confocal with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which pass through the point (x, y)] and putting $c^2=a^2-b^2$, we shall have $x^2+y^2+c^2=\mu^2+\nu^2$, and $cx=\mu\nu$; so that the proposed integral becomes

$$\iint \frac{dx\,dy}{\mu^2-\nu^2}.$$

Now the element of the area is, in elliptic coordinates,

$$dS=\frac{\mu^2-\nu^2}{\sqrt{(\mu^2-c^2)(c^2-\nu^2)}}\,d\mu\,d\nu.$$

Hence the proposed integral, extended over a quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, becomes

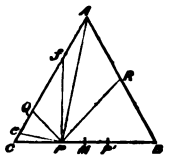
$$\int_c^a\int_0^c \frac{d\mu\,d\nu}{\sqrt{(\mu^2-c^2)(c^2-\nu^2)}}=\frac{\pi}{4}\log\frac{a+b}{a-b},$$

and the required value is $\pi\log\frac{a+b}{a-b}$.

2561. (Proposed by Captain CLARKE, R.E., F.R.S.)—1. A point is taken at random on each side of an equilateral triangle; show that the chance that these points form an acute-angled triangle is $1-\frac{2}{\pi}\log(\frac{27}{4}e)$.

2. Find also the chance that a circle passing through the three points cuts each side of the triangle (not produced) in a second point.

Solution by the PROPOSER.



1. First, to determine the chance that the angle at the point P on BC is an obtuse angle. Let P be between C and M, the middle point of CB. Join PA, draw P*e* perpendicular to PA, meeting AC in *e*; also draw P*f* perpendicular to BC; also, Q being any point in *ef*, join PQ, and draw PR perpendicular to it, meeting AB in R. It is then clear that the points P, Q, and any point in RB, make a triangle obtuse-angled at P. Any point in C*e*, with any point in AB, also subtend an obtuse angle at P, but no point in *f*A will do so. Let CQ$=y$, CP$=x$; then, the angle CPQ being the complement of RPB, it is easy to shew that, if CB$=1$, then

$$\mathrm{RB}=\frac{(1-x)(2x-y)}{x+y};$$

and supposing R to coincide with A, we derive for C*e* the value

$$Ce=y_1=2x+3-\frac{6}{2-x},\quad \text{also}\quad Cf=y_2=2x.$$

The total number of ways, then, in which points on AB, AC can be combined with P so as to form a triangle obtuse-angled at P is

$$2\int_0^{\frac{1}{2}}\left\{y_1+\int(\mathrm{RB})\,dy\right\}dx = 2\int_0^{\frac{1}{2}}\left(y_1+\int_{y_1}^{y_2}\left\{\frac{3x(1-x)}{x+y}-1+x\right\}dy\right)dx$$

$$= 2\int^{\frac{1}{2}}\left\{y_1+3x(1-x)\log\frac{x+y_2}{x+y_1}-(1-x)(y_2-y_1)\right\}dx$$

$$= 2\int_0^{\frac{1}{2}}\left\{3x(1-x)\log\frac{2-x}{1-x}-x\right\}dx = \tfrac{3}{4}+\tfrac{9}{8}\log\tfrac{3}{4},$$

which is the chance that the triangle formed by the three random points is obtuse-angled at P. The chance, then, that *none* of the angles are obtuse is $1-\frac{9}{8}\log(\frac{3}{64}\epsilon)$.

It is interesting, in connection with this, to remark that, P, Q, R being random points on the sides, the chance that the triangle formed by AP, BQ, CR is acute-angled is $1-3\log(\frac{3}{64}\epsilon)$. For we may show that, if QB and CR intersect on the semicircle whose diameter is CB, the relation between $AQ=u$ and $AR=v$ is $1-2(u+v)+uv=0$. Supposing, then, the point of intersection O to be without the semicircle, the triangle is obtuse-angled at O; and the number of ways in which PQR can be taken to produce this result is

$$\int_0^{\frac{1}{2}} v\,du = \int_0^{\frac{1}{2}}\left(2-\frac{3}{2-u}\right)du = 1+3\log\tfrac{3}{4} = \log(\tfrac{3}{64}\epsilon),$$

consequently the number of ways in which the points can be taken so as to produce an acute-angled triangle is $1-3\log(\frac{3}{64}\epsilon)$.

2. We shall first determine the number of circles which do not cut AB and AC twice. Let R, Q be any two points on these sides, and let the circle passing through A, R, Q cut BC in P, P′: also let $CQ=y$, $BR=z$, $CP=x$, $CP'=x'$, then we have

$$xx'=y,\quad (1-x)(1-x')=z,\quad x+x'=1+y-z;$$

therefore
$$x'-x=\left\{(1+y-z)^2-4y\right\}^{\frac{1}{2}},$$

which is the value of PP′. When this vanishes, $z=(1-y^{\frac{1}{2}})^2$. Now QR and any point between P and P′ determine a circle which encloses A, that is, does not again cut AB and AC. The total number of circles therefore fulfilling this condition is

$$\int_0^1\int_0^{(1-y^{\frac{1}{2}})^2}\left\{(1+y-z)^2-4y\right\}^{\frac{1}{2}}dy\,dz = \int_0^1\left\{y\log y+\tfrac{1}{2}(1-y)^2\right\}dy = \tfrac{1}{12}.$$

Therefore the number of circles which enclose either A, B, or C is $\frac{1}{4}$; or the chance that the circle cuts each side in a second point is $\frac{3}{4}$.

2620. (Proposed by G. M. Smith, B.A.)—1. A rigid body is rotating round a fixed axis; suddenly this axis becomes free, and a parallel axis becomes fixed elsewhere in the body. Find the new initial angular velocity, and the locus of the new axis if the angular velocity is unchanged.

2. A right prism, whose base is a regular polygon of n sides, falls (without sliding) down a rough inclined plane, the contact being supposed to take place only at the corners; find its angular velocity just before the mth impact.

3. Hence deduce the law of motion of a cylinder or sphere down a rough inclined plane.

Solution by the Rev. R. TOWNSEND, F.R.S.; *the* PROPOSER; *and others.*

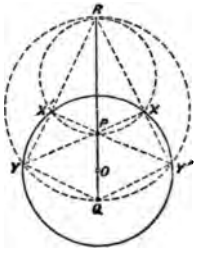
1. There being no change of *vis viva* under the circumstances supposed, we have $\omega^2 I = \omega'^2 I$, ω and ω' being the angular velocities, and I and I′ the moments of inertia round the new and old axes. If $\omega' = \omega$, then $I' = I$, and the locus in question is consequently a cylinder of revolution round the parallel axis passing through the centre of gravity of the body.

2. If M be the mass of the prism, I its moment of inertia round any one of its edges, a the length of any side of the polygon, and i the inclination of the plane to the horizon; then, the *vis viva* of the body being due entirely to the action of gravity, we have, just before the mth impact $\omega^2 I = 2Mg \cdot ma \sin i$, the centre of gravity of the body having been lowered through the height $ma \sin i$, since the commencement of the motion.

3. If r be the radius of the cylinder or sphere, and v the velocity of its centre of gravity after rolling from rest through the space s on the plane; since then I, for the former $= \frac{1}{2}Mr^2$, and for the latter $= \frac{2}{3}Mr^2$, and since v^2 for both $= \omega^2 r^2$, therefore v^2 for both $= 2fs$, or the motion for both is uniform acceleration; f being $= \frac{1}{2}g \sin i$ for the former, and $= \frac{2}{3}g \sin i$ for the latter.

2626. (Proposed by J. J. WALKER, M.A.)—To draw through a given point within a circle a chord which shall be divided at the point in extreme and mean ratio.

Solution by the Rev. R. TOWNSEND, F.R.S.; *the* PROPOSER; *and others.*



The following construction will place the two chords, when real, in position, whatever be the ratio of their section, viz., if O be the centre of the given circle, P the given point, Q the point on the diameter passing through it for which OQ = OP, and R the point of external section of QP in the given ratio; then the two circles on PR and QR as diameters will evidently intersect the circle at the two pairs of extremities X and Y, X′ and Y′ of the two chords required.

This is evident from the parallelism of the two pairs of lines PX and QY′, PX′ and QY, and from the equality of the two pairs PY and QY′, PY′ and QY.

2532. (Proposed by W. S. B. WOOLHOUSE, F.R.S.)—If within a given enclosed area three points be taken at random as the apices of a triangle, the average of the square of the area of all such triangles will be reduced to one-third the value if one of the points be fixed at the centre. Also if within a given volume of space four points be taken at random as the apices of a tetrahedron, the average of the square of the volume of all such tetrahedra will be reduced to one-fourth the value if one point be fixed at the centre. Show that this theorem is true when the enclosed area or volume of space is *of any form whatever*.

2514. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Let a given volume of space, having a convex boundary *of any form whatever*, be referred to its centre of gravity and the principal axes of rotation, and let M denote the total volume and h, k, l what may be called compressed radii of gyration round the axes, so that

$$M.h^2 = \iiint x^2\,dx\,dy\,dz, \quad M.k^2 = \iiint y^2\,dx\,dy\,dz, \quad M.l^2 = \iiint z^2\,dx\,dy\,dz;$$

then

1. If *four* points be taken at random within the given volume, the average square of the volume of the tetrahedron which connects them is $\frac{1}{3}h^2k^2l^2$.

2. If *six* points be taken at random, within the given volume, as the corners of a hexaedron, the probability that two of the points shall be reentrant is $\dfrac{10l^2k^2l^2}{M^2}$.

2556. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Four solids, *having any forms whatever*, being given, show how to determine sets of six points fixed in each, such that if the four solids be placed in any positions in space and four points be taken arbitrarily, one in each solid, as the apices of a tetrahedron, the average square of volume in respect of all such tetrahedra, shall always be equal to that of the 1296 tetrahedra formed by connecting all the groups obtainable from the four invariable sets of six points.

Solution by the PROPOSER.

To begin with the first case of Quest. 2532, suppose (x_1, y_1), (x_2, y_2), (x_3, y_3) to be the coordinates of the three points referred to rectangular axes, let M denote the magnitude of the given enclosed area; dm_1, dm_2, dm_3 differential elements of the area in the immediate vicinity of the respective points; and Δ the area of the variable triangle. Then

$$2\Delta = x_3(y_1 - y_2) - y_3(x_1 - x_2) + x_1y_2 - x_2y_1,$$

$$\therefore\ 4\Delta^2 = x_3^2(y_1 - y_2)^2 + y_3^2(x_1 - x_2)^2 + (x_1y_2 - x_2y_1)^2 + Px_3 + Qy_3 + Rx_3y_3,$$

where P, Q, R are functions of (x_1, y_1), (x_2, y_2), and independent of (x_3, y_3).

Now the average value of Δ^2 is $= \iiint \frac{dm_1}{M}\frac{dm_2}{M}\frac{dm_3}{M}\Delta^2$; and if the axes of coordinates be the principal axes through the centre of gravity of the given area, and h, k denote the radii of gyration of the area round those axes, we shall have generally

$$\int \frac{dm}{M}x^2 = h^2, \int \frac{dm}{M}y^2 = k^2; \int \frac{dm}{M}xy = 0, \int \frac{dm}{M}x = 0, \int \frac{dm}{M}y = 0.$$

Hence we find

$$4\int \frac{dm_3}{M}\Delta^2 - h^2(y_1^2 - 2y_1y_2 + y_2^2) + k^2(x_1^2 - 2x_1x_2 + x_2^2) + (x_1^2y_2^2 - 2x_1y_1x_2y_2 + x_2^2y_1^2).$$

Similarly, with the next integration,

$$4\iint \frac{dm_2}{M}\frac{dm_3}{M}\Delta^2 = h^2(y_1^{2}{}^{*}+k^2)+k^2(x_1^{2}{}^{*}+h^2)+(k^2x_1^{2}{}^{*}+h^2y_1^{2}),$$

or
$$\iint \frac{dm_2}{M}\frac{dm_3}{M}\Delta^2 = \frac{h^2k^2}{2}\left(\frac{x_1^2}{h^2}+\frac{y_1^2}{k^2}+1\right)\ldots\ldots\ldots\ldots(\alpha);$$

therefore
$$\iiint \frac{dm_1}{M}\frac{dm_2}{M}\frac{dm_3}{M} = \frac{h^2k^2}{2}(1+1+1) = \tfrac{3}{2}h^2k^2 \ \ldots\ldots(\beta),$$

which is the average value of the square of the area of the triangle when the three points are taken arbitrarily within the given surface. The expression (α) exhibits the same when one of the points (x_1, y_1) is fixed, and this value is unchanged when the point is supposed to be taken anywhere in the periphery of an ellipse concentric with the given area. When the point (x_1, y_1) is fixed at the centre, or origin of coordinates, the value becomes $\frac{1}{4}h^2k^2$, which is one-third of the general average, as was required to be shown.

In the case of three dimensions, if V denote the volume of the variable tetrahedron,

$$3V = z_1\,.\,\Delta(2,3,4)-z_2\,.\,\Delta(3,4,1)+z_3\,.\,\Delta(4,1,2)-z_4\,.\,\Delta(1,2,3),$$

wherein $\Delta(2,3,4)$, $\Delta(3,4,1)$, &c. denote the areas of triangles determined by orthogonal projection of the several points on the plane of xy. This formula was established in one of my early papers "On the Computation of Superficies and Solids," contained in Leybourne's *Mathematical Repository*, Vol. VI., and equivalent expressions may now be found in most works on Analytical Geometry. The algebraic sign of the area thus determined is dependent upon the order in which the four points are taken; but as we are about to deal with the square of the area, this result is always positive, and is determined by the formula

$$9V^2 = z_1^2(\Delta 234)^2+z_2^2(\Delta 341)^2+z_3^2(\Delta 412)^2+z_4^2(\Delta 123)^2+\mathrm{P}z_1z_2+\&c.$$

The axes of coordinates being, as before, the principal central axes, we shall, in integrating with respect to any variable point, have

$$\int\frac{dm}{M}x^2 = h^2, \quad \int\frac{dm}{M}y^2 = k^2, \quad \int\frac{dm}{M}z^2 = l^2 \ \ldots\ldots\ldots(1),$$

$$\int\frac{dm}{M}yz = 0, \quad \int\frac{dm}{M}zx = 0, \quad \int\frac{dm}{M}xy = 0\ldots\ldots\ldots(2),$$

$$\int\frac{dm}{M}x = 0, \quad \int\frac{dm}{M}y = 0, \quad \int\frac{dm}{M}z = 0 \ \ldots\ldots\ldots(3).$$

In operating upon the expression just given for $9V^2$, the six terms $\mathrm{P}z_1z_2+\&c.$, involving binary products of $z_1z_2z_3z_4$, must by virtue of (3) all disappear after any three integrations.

By (β), $\iiint \frac{dm_2}{M}\frac{dm_3}{M}\frac{dm_4}{M}z_1^2(\Delta 234)^2 = \frac{3}{2}h^2k^2z_1^2.$

By (α),
$$\left\{\begin{array}{l}\iiint \frac{dm_2}{M}\frac{dm_3}{M}\frac{dm_4}{M}z_2^2(\Delta 341)^2\\ \iiint \frac{dm_2}{M}\frac{dm_3}{M}\frac{dm_4}{M}z_3^2(\Delta 412)^2\\ \iiint \frac{dm_2}{M}\frac{dm_3}{M}\frac{dm_4}{M}z_4^2(\Delta 123)^2\end{array}\right\} \text{ each} = \frac{h^2k^2}{2}\left(\frac{x_1^2}{h^2}+\frac{y_1^2}{k^2}+1\right).l^2,$$

and, as has been remarked, all subsequent terms = 0. Hence we get

$$9\iiint \frac{dm_2}{M}\frac{dm_3}{M}\frac{dm_4}{M}V^2 = \tfrac{3}{2}h^2k^2z_1^2 + \tfrac{3}{2}h^2k^2l^2\left(\frac{x_1^2}{h^2}+\frac{y_1^2}{k^2}+1\right),$$

$$\therefore \quad \iiint \frac{dm_2}{M}\frac{dm_3}{M}\frac{dm_4}{M} = \frac{h^2k^2l^2}{6}\left(\frac{x_1^2}{h^2}+\frac{y_1^2}{k^2}+\frac{z_1^2}{l^2}+1\right) \ldots\ldots\ldots(\gamma),$$

$$\iiiint \frac{dm_1}{M}\frac{dm_2}{M}\frac{dm_3}{M}\frac{dm_4}{M} = \frac{h^2k^2l^2}{6}(1+1+1+1) = \tfrac{2}{3}h^2k^2l^2 \ldots.(\delta),$$

which is, therefore, the average value of the square of the volume of a tetrahedron connecting four arbitrary points. The formula (γ) expresses the value when one of the points (x_1, y_1, z_1) is fixed, and it has a constant value when that point is taken in the surface of a determinate concentric ellipsoid. When the immoveable point (x_1, y_1, z_1) is at the centre, the value is $\frac{1}{6}h^2k^2l^2$ and is one-fourth of the general average (δ).

The reasoning obviously applies to all boundaries, whether continuous or not, regular or irregular.

The formula (δ) establishes the first part of Quest. 2514; and the second part of the same question is readily deduced from it as follows:—If v denote the volume of the tetrahedron formed by connecting four of the six points, the separate and independent probabilities that the fifth and sixth points shall fall within v are each $\frac{v}{M}$, and the joint probability that they shall both fall within v is $\frac{v^2}{M^2}$. Hence, when the several points take all positions within the given volume of space, the probability that a specified pair of points shall be both reentrant is $\frac{2}{3}\frac{h^2k^2l^2}{M^2}$; and as the same identical amount of probability attaches to each of the fifteen pairs of the six points, it follows that the complete probability of one or other of the several pairs being reentrant is

$$\frac{2}{3}\frac{h^2k^2l^2}{M^2}\times 15 = \frac{10h^2k^2l^2}{M^2}.$$

The theorem stated in Quest. 2556 may be shown to be true more generally, when in place of the square of the volume of the tetrahedron we substitute any arbitrary algebraic function which involves any one set of coordinates to the second degree. The degree of this function will not be changed by altering the position of the axes of coordinates. It will be convenient if these axes be the principal axes through the centre of gravity of the volume of space in which the point $x_4y_4z_4$ is to be taken. Let the proposed function, after the coordinates are transferred to these particular axes, be

$$U = Ax^2 + By^2 + Cz^2 + ayz + bzx + cxy + \alpha x + \beta y + \gamma z + G,$$

where A, B, C, a, b, c, α, β, γ, G are functions of $x_1y_1z_1$, $x_2y_2z_2$, $x_3y_3z_3$, involving each set of coordinates to the second degree. Then the average value of U with respect to the variable point $x_4y_4z_4$ is evidently

$$(U) = \int \frac{dm_4}{M}U = Ah^2 + Bk^2 + Cl^2 + G \ldots\ldots\ldots\ldots\ldots(\epsilon).$$

But for certain special values of $x_4y_4z_4$ we have

$$\begin{aligned}
U_{h\sqrt{3},\,0,\,0} &= 3Ah^2 + ah\sqrt{3} + G,\\
U_{-h\sqrt{3}\;0,\,0} &= 3Ah^2 - ah\sqrt{3} + G,\\
U_{0,\,k\sqrt{3},\,0} &= 3Bk^2 + \beta k\sqrt{3} + G,\\
U_{0,\,-k\sqrt{3},\,0} &= 3Bk^2 - \beta k\sqrt{3} + G,\\
U_{0,0,\,l\sqrt{3}} &= 3Cl^2 + \gamma l\sqrt{3} + G,\\
U_{0,\;,-l\sqrt{3}} &= 3Cl^2 - \gamma l\sqrt{3} + G,
\end{aligned}$$

$\therefore$ (U) =

$$\frac{1}{6}\left\{U_{h\sqrt{3},\,0,\,0} + U_{-h\sqrt{3},\,0,\,0} + U_{0,\,k\sqrt{3},\,0} + U_{0,\,-k\sqrt{3},\,0} + U_{0,\,0,\,l\sqrt{3}} + U_{0,\,0,\,-l\sqrt{3}}\right\}$$

That is, the average value of the function, when $x_4y_4z_4$ takes all the positions assigned to it, is equal to the average value when $x_4y_4z_4$ successively occupies the six determinate points $\pm h\sqrt{3}, 0, 0$; $0, \pm k\sqrt{3}, 0$; $0, 0, \pm l\sqrt{3}$. Now a concentric and coaxial rectangular parallelepiped, the semi-sides of which are $h\sqrt{3}$, $k\sqrt{3}$, $l\sqrt{3}$, has the same radii of gyration (h, k, l) as the solid in which $x_4y_4z_4$ is taken; and therefore, by (ϵ), the average value (U) would be the same if the point $x_4y_4z_4$ were taken either arbitrarily within the parallelepiped or upon each of the six determinate points before mentioned which are obviously the six centres of its six rectangular faces.

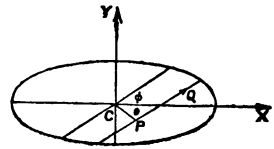
By similarly transferring the coordinates to the principal axes of the other three solids, it may be shown that in each case the average value of the function U is unchanged in magnitude if the point, instead of traversing the volume of the solid, is made to successively occupy the six centres of the faces of the dynamically equivalent parallelepiped. Hence by superposition we arrive at the stated theorem, that when the four points vary throughout the respective solids, placed in any positions, the average value of the function is the same as the average of the 1296 values deduced by associating all the groups obtainable from these invariable sets of six points.

An analogous theorem also exists with respect to certain determinate sets of *four* points in each solid. In fact it is generally true for any determinate sets of points which, regarded as material systems, have the same principal axes and radii of gyration as the respective solids. It might be shown, however, that the conditions here stated admit of being realized only when the number of points in each set is either 4, 6, 8, 12, or 20; that is, the number of corners of one or other of the five regular polyhedra.

NOTE.—It will be observed that the squares of the radii of gyration of the given volume of space round the coordinate axes are respectively equal to $k^2 + l^2$, $l^2 + h^2$, $h^2 + k^2$. The symbols h, k, l are what in Quest. 2514 I have designated as compressed radii of gyration, and, in fact, represent what the several radii would become if the total volume of the body were compressed into one of the three coordinate planes.

Solution of Question 2532, *by the* Rev. J. WOLSTENHOLME, M.A.

Let C be the centre of gravity of any lamina; CX, CY the principal axes of inertia in the plane; and let the area be unity. Then the average value of the square of the triangle PQR is



$$\tfrac{1}{4}\iiint PQ^2 \cdot y^2 \cdot dp\, dq\, dr,$$

y being the distance of R from PQ, and dp, dq, dr elementary areas about PQR.

But $\int y^2\,dr$ = moment of inertia about the line PQ
$= a^2\cos^2\phi + b^2\sin^2\phi + \mathrm{CP}^2\sin^2(\theta+\phi)$;

a, b being radii of gyration about CX, CY; and θ, ϕ the angles which CP and PQ make with CX; therefore the average value

$= \frac{1}{A}\iint r^2\left\{a^2\cos^2\phi + b^2\sin^2\phi + \mathrm{CP}^2\sin^2(\theta+\phi)\right\}dp\,dq$, $(\mathrm{PQ}\equiv r)$.

But $\int r^2\cos^2\phi\,dq$ = moment of inertia about a st. line through P parallel to CY
$= b^2 + \mathrm{CP}^2\cos^2\theta$.

So $\int r^2\sin^2\phi\,dq = a^2 + \mathrm{CP}^2\sin^2\theta$, and $\int r^2\sin^2(\theta+\phi)\,dq = a^2\cos^2\theta + b^2\sin^2\theta$,

or average value $= \frac{1}{A}\int(2a^2b^2 + 2a^2\rho^2\cos^2\theta + 2b^2\rho^2\sin^2\theta)\,dp$, $(\mathrm{CP}\equiv\rho)$;

and $\int\rho^2\cos^2\theta\,dp$ = moment of inertia about CY $= b^2$; so $\int\rho^2\sin^2\theta\,dp = a^2$;

therefore, finally, the average value of the square of the triangle $= \frac{3}{2}a^2b^2$.

But average value of square of the triangle CPQ $= \frac{1}{A}\iint \mathrm{CP}^2y'^2\,dp\,dq$ (y' being distance of Q from CP),

and $\int y'^2\,dq$ = moment of inertia about CP $= a^2\cos^2\theta + b^2\sin^2\theta$,

therefore the average value $= \frac{1}{A}\int(a^2\rho^2\cos^2\theta + b^2\rho^2\sin^2\theta)\,d\theta$
$= \frac{1}{2}a^2b^2$ (for the same reason as before).

The investigation for three dimensions is of much the same kind, referring to the principal axes of inertia at the centre of gravity; and the corresponding results are $\frac{2}{3}a^2b^2c^2$ and $\frac{1}{6}a^2b^2c^2$; b^2+c^2, c^2+a^2, a^2+b^2 being the squares of the radii of gyration about the principal axes.

2348. (Proposed by T. Cotterill, M.A.)—In the system of conics touching two lines at fixed points, the focal cubic cuts a conic in points such that its normals at these points pass through the point of intersection of the two lines.

Solution by J. Dale; H. Tomlinson; *and others.*

Referring the conics to the triangle formed by the two lines and the chord of contact, the equation to any one will be of the form $ux^2 + 2u'yz = 0$, and the coordinates (x, y, z) of a focus will be given by the equations

$$u'(by+cz)^2 + 2ubcx^2 + k = 0,$$
$$u'a^2y^2 - 2ucy(cz+ax) + k = 0,$$
$$u'a^2z^2 - 2ubz(ax+by) + k = 0,$$

the coordinates being trilinear. Eliminating u', u, k; dividing by $ax+by+cz$, and reducing, we get for the equation to the locus of the foci

$$x(y^2-z^2) + 2(y\cos\mathrm{B} - z\cos\mathrm{C})yz = 0.$$

This cubic meets any conic of the system $ux^2 + 2u'yz = 0$ in the points where $x = 0$ meets the conic; that is, in the points of contact B, C; and also in the points where the conic meets the curve

$$u'(y^2-z^2) + ux(z\cos\mathrm{C} - y\cos\mathrm{B}) = 0.$$

But this curve is an equilateral hyperbola, passing through A ($y=0$, $z=0$), and through the centre of $ux^2+2u'yz=0$, and having its axes parallel to the axes of the conic; therefore the normals at the points of intersection pass through A.

Cor.—The focal cubic has a double point at A, the tangents being $y-z=0$ and $y+z=0$, and cuts BC in the foot of the perpendicular from A. Every hyperbola of the system cuts BC in the points ($x=0$, $y-z=0$) and ($x=0$, $y+z=0$).

2364. (Proposed by S. Watson.)—Through each pair of the angles of a triangle circles are described so that the internal segments (in reference to the triangle) are similar. Find the locus of the intersection of the radical axes of the three circles.

Solution by J. Dale; *the* Proposer; *and others.*

Let each of the segments subtend an angle 2θ at the corresponding centre; then, the given triangle being taken as that of reference, the trilinear co-ordinates (x, y, z) of the centres O_1, O_2, O_3 are respectively

$$\{-\cos\theta, \cos(C-\theta), \cos(B-\theta)\}, \quad \{\cos(C-\theta), -\cos\theta, \cos(A-\theta)\},$$
$$\{\cos(B-\theta), \cos(A-\theta), -\cos\theta\}.$$

The radical axes are the straight lines through A, B, C, perpendicular to O_2O_3, O_3O_1, O_1O_2, respectively. The equation to O_2O_3 is

$$\{\cos^2\theta-\cos^2(A-\theta)\}x+\{\cos(A-\theta)\cos(B-\theta)+\cos(C-\theta)\cos\theta\}y$$
$$+\{\cos(B-\theta).....\}z=0;$$

the equation to the perpendicular to O_2O_3 is $y\sin(B+\theta)-z\sin(C+\theta)=0$;
the equation to the perpendicular to O_3O_1 is $z\sin(C+\theta)-x\sin(A+\theta)=0$;
the equation to the perpendicular to O_1O_2 is $x\sin(A+\theta)-y\sin(B+\theta)=0$.

Eliminating θ, we find for the equation to the locus

$$yz\sin(B-C)+zx\sin(C-A)+xy\sin(A-B)=0,$$

which represents an equilateral hyperbola, circumscribing the triangle and passing through its centre of gravity, since the equation is satisfied by

$$x\sin A=y\sin B=z\sin C.$$

The centre (ξ, η, ζ) of the hyperbola is given by the equations

$$\frac{\xi}{\sin A\sin^2(B-C)}=\frac{\eta}{\sin B\sin^2(C-A)}=\frac{\zeta}{\sin C\sin^2(A-B)},$$

therefore

$$(a\xi)^{\frac{1}{2}}+(b\eta)^{\frac{1}{2}}+(c\zeta)^{\frac{1}{2}}=0,$$

that is, the centre lies on the maximum inscribed ellipse. The straight lines AO_1, BO_2, CO_3 intersect in the point $x\cos(A-\theta)=y\cos(B-\theta)=z\cos(C-\theta)$, and this point also lies on the hyperbola [a property which was given by the Editor as Quest. 1281, in the *Educational Times* for July, 1862].

If O_4, O_5, O_6 are the centres of circles whose external segments are equal to the above internal segments, then the radical axes of these circles intersect in the point $x\cos(A+\theta) = y\cos(B+\theta) = z\cos(C+\theta)$, which lies on the hyperbola, and the lines AO_4, BO_5, CO_6 intersect in the point $x\sin(A-\theta) = y\sin(B-\theta) = z\sin(C-\theta)$, which also lies on the hyperbola.

The four points $x\sin(A+\theta)=$&c., $x\cos(A+\theta)=$&c., $x\cos(A-\theta)=$&c., $x\sin(A-\theta)=$&c., form a quadrilateral whose diagonals intersect in the points

$$\frac{x}{\sin A} = \frac{y}{\sin B} = \frac{z}{\sin C}, \quad \frac{x}{\cos A} = \frac{y}{\cos B} = \frac{z}{\cos C};$$

$$\frac{x}{\sin(B-C)} = \frac{y}{\sin(C-A)} = \frac{z}{\sin(A-B)}.$$

2430. (Proposed by the Rev. R. H. WRIGHT, M.A.)—A cycloid revolving about its axis generates a solid which moves in the direction of the axis through a fluid with a given velocity; compare the resistances in the two cases when the vertex and the base are respectively foremost.

Solution by J. DALE; J. J. WALKER, M.A.; E. MCCORMICK; *and others.*

Let ρ = density of fluid, v = velocity; then, when the vertex is first,

$$\text{whole resistance } (R_1) = \pi\rho v^2 \int y \left(\frac{dy}{ds}\right)^2 dy.$$

By the nature of the cycloid $x = a(1-\cos\theta)$, $y = a(\theta+\sin\theta)$;

therefore $R_1 = 2\pi\rho v^2 a^2 \int_0^\pi (\theta+\sin\theta)\cos^4\tfrac{1}{2}\theta\, d\theta = \tfrac{1}{24}\pi\rho v^2 a^2 (9\pi^2-16)$.

When the base is foremost, resistance $(R_2) = \tfrac{1}{2}\pi^3\rho v^2 a^2$;

therefore $R_1 : R_2 = 9\pi^2-16 : 12\pi^2$.

2548. (Proposed by J. J WALKER, M.A.) — Given $\cos^2 x\cos^2 a = \sin^2 a\cos^2 a - 4\cos x\sin(x-a)\sin^3 a$; it is required to find the angle x.

Solution by the PROPOSER.

Put $x = \theta + a$, and develope; then

$$\cos^2\theta\cos^4 a - 2\sin\theta\cos\theta\sin a\cos^3 a$$
$$= \cos^2\theta\sin^2 a\cos^2 a - 4\sin\theta\cos\theta\sin^3 a\cos a + 4\sin^2\theta\sin^4 a, \quad]$$

or $\cos\theta\,(\cos\theta\cos\alpha - 2\sin\theta\sin\alpha)\cos^3\alpha = (\cos\theta\cos\alpha - 2\sin\theta\sin\alpha)^2\sin^3\alpha$,
whence either $\cos\theta\cos\alpha - 2\sin\theta\sin\alpha = 0$ .....(1),
or else $\cos\theta\cos^3\alpha = (\cos\theta\cos\alpha - 2\sin\theta\sin\alpha)\sin^2\alpha$ ....(2).
From (1), $\cos(\theta+\alpha) = \sin\theta\sin\alpha$, or $\cos x = \sin(x-\alpha)\sin\alpha$,

whence $$\tan x = \frac{1+\sin^2\alpha}{\sin\alpha\cos\alpha} = 2\tan\alpha + \cot\alpha.$$

From (2), $\cos^2\alpha\cos(\theta+\alpha) = \sin^2\alpha\cos(\theta+\alpha) - \sin\theta\sin\alpha$,
or $\cos x\,(\sin^2\alpha - \cos^2\alpha) = \sin(x-\alpha)\sin\alpha = \sin x\sin\alpha\cos\alpha - \cos x\sin^2\alpha$,

whence $$\tan x = \frac{2\sin^2\alpha - \cos^2\alpha}{\sin\alpha\cos\alpha} = 2\tan\alpha - \cot\alpha.$$

[The equation may be otherwise solved as follows:—
$$\cos^2\alpha\,(\cos^2 x - \sin^2\alpha) + 4\cos x\sin(x-\alpha)\sin^3\alpha = 0,$$
or $\cos^2\alpha\cos(x+\alpha)\cos(x-\alpha) + 4\cos x\sin(x-\alpha)\sin^3\alpha = 0$;
and dividing by $\cos x\cos(x-\alpha)$ and developing, we get
$$\tan^2 x - 4\tan\alpha\tan x + 4\tan^2\alpha = \cot^2\alpha;$$
therefore $$\tan x = 2\tan\alpha \pm \cot\alpha,$$
which agrees with Mr. WALKER's results.]

2426. (Proposed by C. TAYLOR, M.A.)—If F_n denote the fraction $\frac{1+a^2+a^4+\ldots+a^{2n}}{a+a^3+\ldots+a^{2n-1}}$, prove that, in general,

(1) $\ldots F_n > \frac{n+1}{n} + \left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2$, (2) $\ldots F_n > \frac{1}{2}\cdot\frac{n+1}{n}\left(a+\frac{1}{a}\right)$, (3) $\ldots \frac{n}{n+1}F_n$ increases with n.

Solution by the PROPOSER.

It may be observed that the "inductive" arrangement is not essential to the solution of Quest. 1521, given on p. 105 of Vol. VI. of the *Reprint*.

(1). It is there shown that

$$F_n - F_{n+1} < \frac{1}{n(n+1)} < \frac{1}{n} - \frac{1}{n+1} \quad \ldots\ldots\ldots\ldots\ldots (a).$$

By successive applications of (a), we have $F - F_n < 1 - \frac{1}{n}$;

therefore $$F_n > a + \frac{1}{a} - 1 + \frac{1}{n} > \left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 + \frac{n+1}{n};$$

A fortiori, $$F_n > \frac{n+1}{n}.$$

(2) and (3). By (*a*) we have

$$F_n - F_{n+1} < \frac{1}{(n+1)^2} \cdot \frac{n+1}{n} \cdot \frac{1}{(n+1)^2} F_n, \text{ a fortiori.}$$

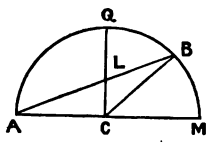
Write $n-1$ for n and transpose; then

$$F_n > \left(1-\frac{1}{n^2}\right) F_{n-1}, \quad \frac{n}{n+1} F_n > \frac{n-1}{n} F_{n-1} > \ldots > \frac{1}{2}\left(a+\frac{1}{a}\right).$$

2584. (Proposed by the late Rev. R. H. WRIGHT, M.A.) — AQM is a semicircle, QC the vertical radius, AB a chord cutting QC in L so that $QC = n \cdot CL$; find the ratio between the triangles BCL and ACL.

Solution by S. WATSON; *the* REV. J. WOLSTENHOLME, M.A.; *and many others.*

1. Draw the lines as in the question; then



$$AL^2 = CQ^2 + CL^2 = (n^2+1)\,CL^2,$$

and $$AL \cdot LB = CQ^2 - CL^2 = (n^2-1)\,CL^2;$$

therefore $$\frac{\text{triangle ACL}}{\text{triangle BCL}} = \frac{AL}{BL} = \frac{n^2+1}{n^2-1}.$$

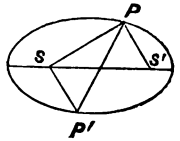
2. *Otherwise:* $$\frac{\Delta BCL}{\Delta ACL} = \frac{BL}{AL} = \frac{AB}{AL} - 1 = \frac{2\cos A}{\sec A} - 1 = \cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A};$$

and $$\frac{CL}{CQ} = \frac{1}{n} = \frac{CL}{CA} = \tan A; \quad \therefore \frac{\Delta ACL}{\Delta BCL} = \frac{n^2+1}{n^2-1}.$$

2585. (Proposed by the Rev. W. ROBERTS, M.A.)—The ends of any diameter of an ellipse are joined with one of the foci; prove that the angles made by the joining lines with the major axis are the eccentric angles corresponding to the ends of a focal chord.

Solution by the REV. JOSEPH WOLSTENHOLME, M.A.

The angle P'SS' = PS'S, therefore the angles are equal to PSS' and PS'S, or θ and ϕ suppose; hence putting 2σ for the perimeter of the triangle SPS' and a for the side SS', we have



$$\tan\frac{\theta}{2}\tan\frac{\phi}{2} = \frac{\sigma-a}{\sigma} = \frac{1-e}{1+e}, \text{ or } \cos\frac{\theta+\phi}{2} = e\cos\frac{\theta-\phi}{2},$$

which is the condition that the chord

$$\frac{x}{a}\cos\frac{\theta+\phi}{2} + \frac{y}{b}\sin\frac{\theta+\phi}{2} = \cos\frac{\theta-\phi}{2},$$

joining the points whose eccentric angles are θ and ϕ, should pass through a focus.

NOTE ON RANDOM LINES. By W. S. B. Woolhouse, F.R.A.S.

The term "random line" is in itself indeterminate as to its interpretation, requiring some sort of restrictive adjunct to impart to it a definite meaning, and this ambiguity naturally arises out of the number of variable elements upon which its position necessarily depends. In considering any problem in chances, relating to random lines, the definition of the lines so designated, or of the mathematical conditions to which they are subject, ought evidently to be elicited from the particular mode in which they are assumed to be generated. There can be no doubt whatever as to the precise signification of a random line drawn from a given point, as such line, for obvious reasons, must be regarded as taking a random direction from the given point, and therefore as equally likely to fall upon equal angular spaces round the same. And just as intelligible and distinct a meaning attaches to a random chord drawn in a given circle, since the mechanical process of its generation, by actual drawing, necessarily implies a commencement at one of its extremities, and this extremity must evidently be a random point on the circumference, the interpretation of which is well understood. In my solution to Quest. 1894 (*Reprint*, Vol. V., p. 110), I have shown at the outset that the conditions which subsist under such circumstances are exactly the same as when the chord is supposed to join two random points taken in the circumference. In each of these cases the definition arrived at is obviously the only one that is compatible with the existing conditions.

Again, the phrases—

1. A random line drawn from a random point,
2. A random line connecting two random points,

are not definite in their meaning until it is stated how, or under what limitations, the random points are supposed to be taken. If they are taken on a given finite area, the two sets of conditions are then essentially different, both in form and result, since if we conceive a random point to be taken as the origin of the line, the line itself may equally proceed in all directions round this origin in the first case, but obviously not so in the second. If, however, the random points are supposed to be taken on an infinitely extended plane, the two cases just stated will then merge into like conditions, and either of them will determine a random line just the same as that recently adopted by Mr. Crofton,* who ingeniously assumes it to be one of an assemblage of lines thus described :—"Divide the angular space round any point into a number of equal infinitesimal angles, and for every direction let the plane be ruled with an infinity of parallel equidistant lines, the common infinitesimal distance being the same for every set of parallels." The perfectly equal distribution over the plane, maintained by this assemblage of lines and their intersections, must necessarily impart an elementary and interesting character to the results of investigations relating to the intersections of random lines, when these are made to depend upon such an uniform combination of materials. The conditions conveyed by these equally distributed combinations are virtually the same as those which subsist in either of the two foregoing cases, provided that the random points which determine the lines are conceived to range absolutely over an infinite plane ; but such conditions as these do not appear to be capable of being suggested or sustained by any recognised mode of generating lines by human or finite agency, unless indeed we might suppose the lines to be material and rigid, and to be arbitrarily thrown upon the plane. The system of random lines thus laid down by Mr. Crofton has however this evident recommendation, that, as an ideal matrix of an arbitrary line on a plane of indefinite extent, it presents the most convenient basis for mathematical deductions.

* See Reprint, Vol. VII., p. 85.

2676. (Proposed by Dr. SALMON, F.R.S.)—Let the sides of the triangle of reference subtend at a point $(\alpha', \beta', \gamma')$ angles $(\theta, \theta', \theta'')$. Then

$$\sin\theta\left(\frac{\alpha}{\alpha'}\right)^{\frac{1}{2}} + \sin\theta'\left(\frac{\beta}{\beta'}\right)^{\frac{1}{2}} + \sin\theta''\left(\frac{\gamma}{\gamma'}\right)^{\frac{1}{2}} = 0$$

is a conic touching the three lines and having $(\alpha', \beta', \gamma')$ for focus. Also

$$\tan\tfrac{1}{2}\theta\left(\frac{\alpha'}{\alpha}\right) + \tan\tfrac{1}{2}\theta'\left(\frac{\beta'}{\beta}\right) + \tan\tfrac{1}{2}\theta''\left(\frac{\gamma'}{\gamma}\right) = 0$$

is a conic circumscribing the triangle and having $(\alpha', \beta', \gamma')$ for focus.

If we denote by λ, μ, ν the perpendiculars drawn from the vertices of the triangle of reference on any tangent, and ρ, ρ', ρ'' the distances of $(\alpha', \beta', \gamma')$ from the vertices; then

$$\sin\theta\left(\frac{\rho}{\lambda}\right) + \sin\theta'\left(\frac{\rho'}{\mu}\right) + \sin\theta''\left(\frac{\rho''}{\nu}\right) = 0$$

is the tangential equation of the first of these conics, and

$$\sin\tfrac{1}{2}\theta\left(\frac{\lambda}{\rho}\right) + \sin\tfrac{1}{2}\theta'\left(\frac{\mu}{\rho'}\right)^{\frac{1}{2}} + \sin\tfrac{1}{2}\theta''\left(\frac{\nu}{\rho''}\right)^{\frac{1}{2}} = 0$$

is the tangential equation of the second.

Solution by W. S. MCCAY, B.A.

If Σ be the tangential equation of a conic whose foci are $(\alpha', \beta', \gamma')$, $(\alpha'', \beta'', \gamma'')$, then, Ω being the tangential equation of the circular points at infinity,

$$\Omega - \Sigma = (\lambda\alpha' + \mu\beta' + \nu\gamma')(\lambda\alpha'' + \mu\beta'' + \nu\gamma'')$$

the foci being intersections of common tangents.

Hence, Σ being $(A, B, C, F, G, H)(\lambda, \mu, \nu)^2$, we have

$$\begin{array}{l|l} \alpha'\alpha'' = 1 - A \ldots(1) & \beta'\gamma'' + \gamma'\beta'' = 2(-F - \cos A)\ldots(4) \\ \beta'\beta'' = 1 - B \ldots(2) & \alpha'\gamma'' + \gamma'\alpha'' = 2(-G - \cos B)\ldots(5) \\ \gamma'\gamma'' = 1 - C \ldots(3) & \alpha'\beta'' + \beta'\alpha'' = 2(-H - \cos C)\ldots(6). \end{array}$$

Eliminating α'', β'', γ'' from the second three by the first three, we get three equations like

$$\beta'^2 + 2\beta'\gamma' \cos A + \gamma'^2 = B\gamma'^2 - 2F\beta'\gamma' + C\beta'^2$$

or

$$\rho^2 \sin^2 A = \beta\gamma'^2 - 2F\beta'\gamma' + C\beta'^2.$$

Now, if the triangle of reference circumscribe the conic, $A = B = C = 0$. Hence (since $a\alpha' = \rho'\rho'' \sin\theta$) we have

$$\begin{aligned} F : G : H &= \rho^2 \sin^2 A \,.\, \alpha' : \rho'^2 \sin^2 B \,.\, \beta' : \rho''^2 \sin^2 C \,.\, \gamma \\ &= \rho \sin A \sin\theta : \rho' \sin B \sin\theta' : \rho'' \sin C \sin\theta''. \end{aligned}$$

Hence

$$\frac{\rho \sin A \sin\theta}{\lambda} + \frac{\rho' \sin B \sin\theta'}{\mu} + \frac{\rho'' \sin C \sin\theta''}{\gamma} = 0,$$

or calling λ, μ, ν the perpendiculars, as Dr. Salmon does,

$$\frac{\rho \sin\theta}{\lambda} + \frac{\rho' \sin\theta'}{\mu} + \frac{\rho'' \sin\theta''}{\gamma} = 0,$$

and of course the first equation follows readily.

Again, if the triangle be inscribed in the conic,

$$\begin{array}{l|l} BC = F^2 & \rho \sin A = \gamma'\sqrt{B} + \beta'\sqrt{C} \\ AC = G^2 & \rho' \sin B = \alpha'\sqrt{C} + \gamma'\sqrt{A} \\ AB = H^2 & \rho'' \sin C = \beta'\sqrt{A} + \alpha'\sqrt{B}. \end{array}$$

Hence
$$\sqrt{A} : \sqrt{B} : \sqrt{C} = \alpha' (-\rho \sin A\alpha' + \rho' \sin B\beta' + \rho' \sin C\gamma')$$
$$: \beta' (\rho \sin A\alpha' - \rho' \sin B\beta' + \rho'' \sin C\gamma')$$
$$: \gamma' (\rho \sin A\alpha' + \rho' \sin B\beta' - \rho'' \sin C\gamma'),$$

therefore
$$\sqrt{A} : \sqrt{B} : \sqrt{C} = \alpha' (-\sin\theta + \sin\theta' + \sin\theta'')$$
$$: \beta' (\sin\theta - \sin\theta' + \sin\theta'') : \gamma' (\sin\theta + \sin\theta' - \sin\theta')$$
$$= \alpha' \tan \tfrac{1}{2}\theta : \beta' \tan \tfrac{1}{2}\theta' : \gamma' \tan \tfrac{1}{2}\theta'';$$

and the tangential equation is

$$\sqrt{(\alpha' \tan \tfrac{1}{2}\theta\lambda)} + \sqrt{(\beta \tan \tfrac{1}{2}\theta'\mu)} + \sqrt{(\gamma' \tan \tfrac{1}{2}\theta''\nu)} = 0,$$

which gives Dr. Salmon's second equation, and his fourth comes by putting λa, μb, νc for λ, μ, ν.

In the same way we find the equation, if the triangle be self-conjugate, to be

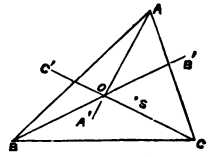
$$\tan\theta \left(\frac{\alpha}{\alpha'}\right)^2 + \tan\theta' \left(\frac{\beta}{\beta'}\right)^2 + \tan\theta'' \left(\frac{\gamma}{\gamma'}\right)^2 = 0.$$

[Dr. SALMON remarks that the theorems are obtained immediately by taking the reciprocals of the ordinary trilinear equations of a circle inscribed in a triangle and circumscribing it, using the principle that the ratio of the distances of any point from a line and from the origin of reciprocation is equal to the corresponding quantity in the reciprocal figure.]

2675. (Proposed by Professor SYLVESTER.)—If two triangles lie in perspective, prove that a cubic curve may be drawn through the six angles, such that for each triangle the tangents at its angles shall all three meet together on the curve.

Solution by F. D. THOMSON, M.A.

Let ABC, A′B′C′ be the two triangles; and let AA′, BB′, CC′ meet in O. Then we can draw a cubic through ABCA′B′C′O so as to satisfy the *two* conditions that the tangents at ABC shall meet in a point (S suppose) which shall be upon the curve.



In order to prove Professor SYLVESTER'S theorem, it will be sufficient to show that the tangents to this cubic at A′B′C′ meet in a point (S′ suppose) upon the curve.

It is convenient (as in the *Messenger of Mathematics*, Vol. III., p. 15) to call the point where the chord joining two given points on a cubic meets the

curve again, the *satellite* of these two points; or, if the two points coincide, the *satellite* of the point of contact of the tangent; *e. g.*, S is the *satellite* of A, B, or C.

Then, as in the paper referred to, since AOA′ is a straight line, the *satellites* of AOA′ are in a straight line.

Let X be the *satellite* of O, a' of A′, b' of B′, c' of C′. Then SXa' is a straight line, and similarly SXb', SXc' are straight lines; therefore $a'b'c'$ coincide in a point (S′ suppose) on the curve.

[In Professor SYLVESTER's system of nomenclature, the Satellite is termed specifically the Connective, generically the Residual, of the two points to which it appertains. The Connective of a reiterated point is its Tangential.]

2646. (Proposed by Capt. A. R. CLARKE, R.E., F.R.S.)—A cube being cut by any plane, what is the chance that the section is a hexagon?

Solution by the PROPOSER.

Let the origin of coordinates be at the centre of the cube and the axes parallel to its sides. Imagine a sphere of radius indefinitely greater than the side of the cube, concentric with the cube. Let the positive directions of the axes of coordinates meet the sphere in X, Y, Z. A little consideration will show that we may limit the cutting planes under consideration to such as have their normals falling within the triangle XYZ. Further, if U, V, W be the middle points of ZY, XZ, YX, and the arcs XU, YV, ZW meet in O, it will be seen that we may limit the cutting planes to such as have their normals falling within any one of the six equal triangles into which XYZ is divided. We shall consider then only such cutting planes as have their normals falling within OVZ. Let e be that angle of the cube which is on the line joining the centre of the cube with O; a, b, c the angles contiguous to e, so that ae, be, ce are parallel to the positive directions of the axes; d is the fourth angle of the face eab parallel to the plane of XY. Let any cutting plane (P), whose normal meets the sphere in P (in OVZ), cut the edges ea, eb, ec in f, g, h, then $eg > ef > eh$, so that if the plane do not cut ec it does not cut ea or eb. Let $\alpha, \beta, \gamma, \delta, \epsilon$ be the distances of a, b, c, d, e from a plane parallel to the plane P, and passing through the centre of the cube. Then, if the side of the cube $= 2$, and if l, m, n be direction cosines of the normal to P, we have

$$\alpha = -l+m+n,\ \beta = l-m+n,\ \gamma = l+m-n,\ \delta = -l-m+n,\ \epsilon = l+m+n.$$

Here we see that $\delta = -\gamma$. But if we confine our attention, as we may, to such cutting planes as lie on that side of the centre of the cube which is towards the point P on the sphere, then to get an hexagonal section, γ must be positive, and δ negative. When $\delta = \gamma = 0$, $l+m = n$, and the locus of this equation is the arc of a great circle which joins UV, meeting OZ in T. In order then to have an hexagonal section, the point P must be within the triangle OTV.

Let the great circle through Z and P cut TV, OV in p and q, and let $ZP = \phi$, $VZP = \theta$, $Zp = \phi_1$, $Zq = \phi'$; then $\cot \phi_1 = \cos\theta + \sin\theta$, and $\cot \phi' = \cos\theta$; hence the ratio of the number of hexagonal sections to the

total number of sections, that is, the probability that a chance section is a hexagon, is

$$p = \frac{N}{D}, \text{ where } N = \int_0^{\frac{1}{2}\pi} \int_{\phi_1}^{\psi'} \gamma \sin\phi \, d\theta \, d\psi, \quad D = \int_0^{\frac{1}{2}\pi} \int_0^{\phi'} \epsilon \sin\phi \, d\theta \, d\phi.$$

Now we have, $l = \cos\theta \sin\phi$, $m = \sin\theta \sin\phi$, $n = \cos\phi$; therefore

$$2N = \int_0^{\frac{1}{2}\pi} \int_{\phi_1}^{\phi'} \left[(1 - \cos 2\phi)(\cos\theta + \sin\theta) - 2 \sin\phi \cos\phi \right] d\theta \, d\phi$$

$$= \int_0^{\frac{1}{2}\pi} \left[(\phi' - \phi_1)(\cos\theta + \sin\theta) - \frac{\cos\theta \sin\theta}{1 + \cos^2\theta} \right] d\theta$$

$$= \left[(\phi' - \phi_1)(-\cos\theta + \sin\theta) \right]_0^{\frac{1}{2}\pi} - \int_0^{\frac{1}{2}\pi} \frac{2d\theta}{1 + \cos^2\theta} + \int_0^{\frac{1}{2}\pi} \frac{3d\theta}{2 + \sin 2\theta}$$

$$= \sqrt{3} \cot^{-1}\sqrt{3} - \sqrt{2} \cot^{-1}\sqrt{2};$$

$$2D = \int_0^{\frac{1}{2}\pi} \left[(\cos\theta + \sin\theta)(\phi' - \sin\phi' \cos\phi') - \cos^2\phi' + 1 \right] d\theta$$

$$= \left[\phi'(\sin\theta - \cos\theta) \right]_0^{\frac{1}{2}\pi} = \frac{\pi}{4}.$$

Hence the chance that the section is hexagonal is

$$\frac{\sqrt{3} \cot^{-1}\sqrt{3} - \sqrt{2} \cot^{-1}\sqrt{2}}{\frac{1}{4}\pi}, \text{ which is about } 0{\cdot}01646\ldots.$$

NOTE ON THE SOLUTION OF QUESTION 2483. BY ARTEMAS MARTIN.

Mr. INGLEBY [*Reprint*, Vol. IX., p. 84] finds $65^2 = 33^2 + 56^2 = 25^2 + 60^2$, and then says :—"All other cases of the separation of 65^2 are imaginary."

I desire to add that there are two other possible cases.

Taking $p_3 = 8$, $q_3 = 1$, and $r_3 = 1$, we have $65^2 = 63^2 + 16^2$; and taking $p_4 = 2$, $q_4 = 1$, and $r_4 = 13$, we have $65^2 = 39^2 + 52^2$.

2413. (Proposed by Professor MANNHEIM.)—Three planes at right angles to each other, and intersecting in a point A, touch an ellipsoid whose centre is B. Prove that the connector AB and the normals at the three points of contact are generators of a hyperboloid.

Solution by the REV. J. WOLSTENHOLME, M.A.

Suppo-ing the ellipsoid to rest supported by the planes: one position of equilibrium will be when its centre is vertically above A, since its centre is then

in the highest possible position of its spherical locus. The ellipsoid is then kept at rest by gravity along AB and the pressures of the planes at the points of contact along the normals. Let any straight line be drawn meeting the three normals, and take moments of the four forces about it. The moments of the three reactions are zero, because their lines of action intersect it; and the sum of the moments is zero, because there is equilibrium. Hence the moment of gravity is zero; or any straight line drawn intersecting the three normals will intersect AB. The three normals and AB must therefore be generators of the same system of a hyperboloid.

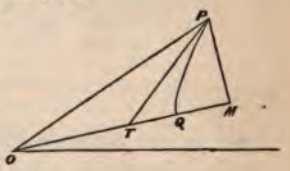
2518. (Proposed by the Rev. W. ROBERTS, M.A.)—1. Let P be a point on a given plane curve (A), O a fixed point, and Q the vertex of an equilateral hyperbola, whose centre is O, and which touches the given curve at P. Prove that the tangent at Q to the curve which is the locus of Q makes with OQ an angle equal to that made with OP by the tangent to (A) at P.

2. The curve (B), which is the inverse of the locus of Q with respect to the origin O, will be reciprocal to the given curve (A); that is to say, if (B) be regarded as the given curve, the original curve (A) will be derived from it in precisely the same manner as it has been derived from (A).

[A conic section referred to its centre, and an ellipse of Cassini, are reciprocal curves of this kind.]

Solution by A. W. PANTON, B.A.

1. Let (r, θ) be the polar coordinates of P, (ρ, ϕ) those of Q, PT the tangent at P, and PM perpendicular on OQ.



Now, by a simple property of the equilateral hyperbola, we have angle MPT = POM. Let $OPT = \epsilon$,

then $\epsilon + 2(\theta - \phi) = 90°$,

and $\cot 2(\theta - \phi) = \tan \epsilon = r\dfrac{d\theta}{dr}$.

Again, from the polar equation of the equilateral hyperbola, $r^2 \cos 2(\theta - \phi) = \rho^2$,

therefore $r \cos 2(\theta - \phi)\, dr - \sin 2(\theta - \phi)\, r^2 (d\theta - d\phi) = \rho d\rho$.

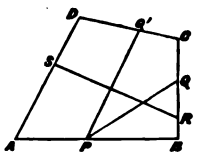
Dividing by $\sin 2(\theta - \phi)$ we have $r^2 dr \dfrac{d\theta}{d\phi} - r^2 (d\theta - d\phi) = \dfrac{\rho d\rho}{\sin 2(\theta - \phi)}$,

whence $\rho^2 d\phi = \rho d\rho \cot 2(\theta - \phi)$; therefore $\rho \dfrac{d\phi}{d\rho} = \cot 2(\theta - \phi) = \tan \epsilon$.

2. Change ρ into ρ^{-1} and r into r^{-1}, then $\rho^2 \cos 2(\theta - \phi) = r^2$, which shows that the inverse of (A) is got from (B) as the inverse of (B) is from (A).

2585. (Proposed by S. WATSON.)—Four points are taken at random on the sides of a quadrilateral; what is the chance of the lines *joining* them intersecting?

Solution by the PROPOSER.



Let ABCD be the quadrilateral, and take first two of the points P, Q on contiguous sides, as AB, BC. Put $AB=a$, $BC=b$, $CD=c$, $DA=d$, $BP=x$, $BQ=y$. Then if R, S be the other points, in order that PQ and RS may intersect, when R lies upon BQ, S must lie on $AP+AD+DC = c+d+a-x$, and when R lies on BP, S must lie on $AD+DC+CQ = c+d+b-y$; hence when P and Q take every position on AB, BC, the number of favourable cases is expressed by

$$\int_0^b \int_0^a \left\{(c+d+a-x)y+(c+d+b-y)x\right\} dy\,dx$$

$$= \tfrac{1}{2}ab\left\{ab+(a+b)(c+d)\right\} \ldots\ldots\ldots\ldots (1).$$

By symmetry, when P and Q lie on BC and CD, on CD and DA, and on DA and AB, the number of favourable positions is respectively

$$\tfrac{1}{2}bc\left\{bc+(b+c)(d+a)\right\}, \qquad \tfrac{1}{2}cd\left\{cd+(c+d)(a+b)\right\},$$

$$\tfrac{1}{2}da\left\{da+(d+a)(b+c)\right\} \ldots\ldots\ldots\ldots (2, 3, 4).$$

Let now Q be on the side opposite to P, as at Q′, and put $CQ'=z$; then according as R lies on BP, BC, CQ′, the positions of S must be $c+d-z$, $a+c+d-x-z$, $d+a-x$; hence the favourable cases, when P and Q′ take every position on AB, CD, are

$$\int_0^c \int_0^a \left\{(c+d-z)x+b(a+c+d-x-z)+(d+a-x)z\right\} dz\,dx$$

$$= \tfrac{1}{2}ac\left\{ac+2bd+(a+c)(b+d)\right\} \ldots\ldots\ldots\ldots (5),$$

and similarly, when P and Q′ lie on BC, DA, we have

$$\tfrac{1}{2}bd\left\{bd+2ac+(a+c)(b+d)\right\} \ldots\ldots\ldots\ldots (6).$$

Now (1)....(6) must be all multiplied by 4, because P and Q, as also R and S, may be interchanged; hence the required chance is

$$p = \frac{4\left\{(1)+(2)\ldots+(6)\right\}}{(a+b+c+d)^4} = \frac{2(ab+ac+ad+bc+bd+cd)^2-8abcd}{(a+b+c+d)^4}.$$

In the case of a parallelogram, we have $c=a$, $d=b$, and then

$$p = \frac{(a^2+4ab+b^2)^2-4a^2b^2}{8(a+b)^4}.$$

In the case of a rhombus, we have $p=\frac{1}{4}$.

2669. (Proposed by J. J. WALKER, M.A.)—If the diagonals d, d' of any plane quadrilateral be divided at their point of intersection in the ratios $p:p'$ and $q:q'$ respectively, and θ be the angle between them, then (1) the square of the third diagonal will be equal to

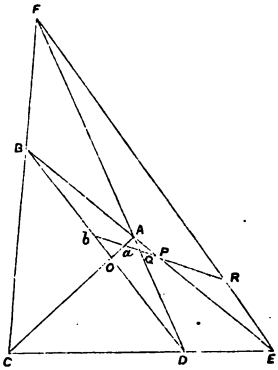
$$\frac{p^2p'^2(q^2-q'^2)^2\,d^2+q^2q'^2(p^2-p'^2)^2\,d'^2+2pp'\,qq'\,(p^2-p'^2)(q^2-q'^2)\,dd'\cos\theta}{(pq-p'q')^2(pq'-p'q)^2};$$

and (2) the line which divides these diagonals in the ratios $m : m'$ and $n : n'$ respectively will divide the third diagonal in the ratio

$$\frac{mnp'q' - m'n'pq}{p'q' - pq} : \frac{m'npq' - mn'p'q}{pq' - p'q}.$$

Solution by S. BILLS; *the* PROPOSER; *and others.*

1. In the accompanying diagram let ABCDEF be a complete quadrilateral; AC, BD, EF being the three diagonals, the first two intersecting in O.



Let $AC = d$ and $BD = d'$; then, by the question, $AO = \frac{dp}{p+p'}$, $CO = \frac{dp'}{p+p'}$, $BO = \frac{d'a}{q+q'}$, $DO = \frac{d'q'}{q+q'}$; also, taking a point a in AC so that $Aa : Ca = m : m'$, and a point b in BD so that $Bb : Db = n : n'$, we shall have

$$Aa = \frac{dm}{m+m'}, \quad Ca = \frac{dm'}{m+m'},$$

$$Bb = \frac{d'n}{n+n'}, \quad Db = \frac{d'n'}{n+n'}.$$

For shortness, put $\frac{p}{p+p'} = u$, $\frac{q}{q+q'} = v$, $\frac{m}{m+m'} = r$, $\frac{n}{n+n'} = s$;

then, taking O for origin, and OA, OB for axes of x and y, the equations of the several lines will be as follows,—

$$AB \ldots\ldots\ldots\ldots\ldots \frac{x}{du} + \frac{y}{d'v} = 1; \quad BC \ldots\ldots -\frac{x}{d(1-v)} + \frac{y}{d'v} = 1;$$

$$CD \ldots\ldots \frac{x}{d(1-u)} + \frac{y}{d'(1-v)} = -1; \quad DA \ldots\ldots \frac{x}{du} - \frac{y}{d'(1-v)} = 1;$$

$$ab \ldots\ldots \frac{x}{d(u-r)} + \frac{y}{d'(v-s)} = 1.$$

Let ba produced meet BA in P, AD in Q, and EF in R. Let (x_E, y_E) be the coordinates of the point E; then we readily find

$$x_E = \frac{du(1-u)}{v-u}, \quad y_E = \frac{d'v(1-v)}{u-v}.$$

Using a similar notation for the other points, we have

$$x_F = \frac{du(1-u)}{1-u-v}, \quad y_F = \frac{d'v(1-v)}{1-u-v}; \quad x_P = \frac{dus(u-r)}{us-vr}, \quad y_P = \frac{d'vr(v-s)}{vr-us};$$

$$x_Q = \frac{du(u-r)(1-s)}{(1-s)u-(1-v)r}, \quad y_Q = \frac{d'(1-v)(v-s)r}{(1-v)r-(1-s)u}.$$

Now $EF^2 = (x_E - x_F)^2 + (y_E - y_F)^2 - 2(x_E - x_F)(y_E - y_F)\cos\theta$ ..... (α).

But $x_E - x_F = du(1-u)\left(\frac{1}{v-u} - \frac{1}{1-u-v}\right) = \frac{dpp'(q'^2-q^2)}{(p'q-pq')(p'q'-pq)}$,

and $y_E - y_F = dv(1-v)\left(\frac{1}{u-v} - \frac{1}{1-u-v}\right) = \frac{d'qq'(p^2-p'^2)}{(p'q-pq')(p'q'-pq)}$.

Substituting the above results in (α), we have

$$EF^2 = \frac{p^2p'^2(q^2-q'^2)^2d^2 + q^2q'^2(p^2-p'^2)^2d'^2 - 2pp'qq'(p^2-p'^2)(q^2-q'^2)\cos\theta}{(pq-p'q')^2(pq'-p'q)^2}$$

2. Again, $AP : EP = x_P - x_A : x_E - x_P$,

and $AQ : FQ = x_Q - x_A : x_F - x_Q$.

But, by a well-known theorem, we have

$ER : FR = EP.AQ : FQ.AP$

$= (x_E - x_P)(x_Q - x_A) : (x_P - x_A)(x_F - x_Q)$ ..... (β).

Substituting (in β) for x_E, x_F, x_P, &c., their values, as given above, we find

$$ER : FR = \frac{m'npq' - mn'p'q}{pq' - p'q} : \frac{mnp'q' - m'n'pq}{p'q' - pq}.$$

If $m = m'$ and $n = n'$, it is evident from the above that $ER = FR$; which shows that the middle points of the three diagonals of a complete quadrilateral lie in a straight line, as is well known from other considerations.

2662. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—Prove that

$$\frac{\sin n\alpha \sin(\beta-\gamma) + \sin n\beta \sin(\gamma-\alpha) + \sin n\gamma \sin(\alpha-\beta)}{\sin(\gamma-\beta) + \sin(\beta-\alpha) + \sin(\alpha-\gamma)}$$

$$= 2\Sigma(pa + q\beta + r\gamma);$$

p, q, r being positive integers whose sum is n, no two being zero, and the coefficient when one is zero being 1 instead of 2.

Solution by MORGAN JENKINS, M.A.

By the ordinary formulæ we have

$$2\sum_{q=1}^{q=n-p-1} \sin\left\{pa + q\beta + (n-p-q)\gamma\right\}$$

$$= \frac{\cos\left\{pa + (n-p)\gamma + \frac{1}{2}(\beta-\gamma)\right\} - \cos\left\{pa + (n-p)\beta - \frac{1}{2}(\beta-\gamma)\right\}}{\sin\frac{1}{2}(\beta-\gamma)}$$

$$= \frac{\cos\left\{pa + (n-p)\gamma + \frac{1}{2}(\beta-\gamma)\right\}}{\sin\frac{1}{2}(\beta-\gamma)} + \text{similar terms formed from}$$

transposition of β and γ.

Again, summing for the variation of p between 1 and $n-2$, we shall obtain for the sum of the series of terms which have the coefficient 2,

$$\frac{\cos(n\alpha-\alpha+\gamma)+\text{sym. terms}-\cos(n\alpha-\alpha+\beta)-\text{sym. terms}}{4\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\gamma-\alpha)} \quad \ldots\ldots (1).$$

Also, summing $\Sigma_{p=1}^{p=n-1} \sin\{p\alpha+(n-p)\beta\}$ and the symmetrical expressions, we shall obtain

$$\frac{-\cos(n\alpha-\alpha+\gamma)-\text{sym. terms}+\cos(n\alpha-\alpha+\beta)+\text{sym. terms}}{4\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\gamma-\alpha)}$$

$$+\frac{\frac{1}{2}\{\cos(n\alpha-\beta+\gamma)-\cos(n\alpha+\beta-\gamma)\}+\text{sym. terms}}{4\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\gamma-\alpha)} \quad \ldots\ldots\ldots (2).$$

Adding (1) to (2), the numerator is equal to the numerator of the given left-hand expression, and the denominator of (1) and (2) to the denominator of the same given expression.

2587. (Proposed by S. ROBERTS, M.A.) — Writing $F(z)$ for $a_0+a_1z+a_1\frac{z^2}{1.2}+\frac{a_3.z^3}{1.2.3}+$ &c., show that the coefficient of $x_1^{k_1}x_2^{k_2}\ldots x_m^{k_m}$ in $F(x_1-x_2).F(x_2-x_3)\ldots.F(x_m-x_1)$ is an invariant, or the leading term of a covariant relative to the function formed by taking the terms of $F(z)$ involved in the coefficient and substituting binomial for exponential coefficients.

Solution by the PROPOSER.

The coefficient of $x^{k_1}\ldots x^{k_m}$ must be formed from the terms of

$$\pm\Sigma\frac{a_{a_1}.a_{a_2}\ldots a_{a_m}}{[a_1][a_2]\ldots[a_m]}(x_1-x_2)^{a_1}(x_2-x_3)^{a_2}\ldots(x_m-x_1)^{a_m},$$

where $a_1+a_2+\ldots+a_m=k_1+k_2+\ldots+k_m$. Suppose, then, we take the product $\pm\frac{a_{a_1}.a_{a_2}\ldots a_{a_m}}{[p_1][a_1-p_1]\ldots[p_m][a_m-p_m]}x_1^{p_1}.x_2^{a_1-p_1}\ldots x_m^{p_m}.x_1^{a_m-p_m}$, we must have

$$p_1+a_m-p_m=k_1,\quad p_2+a_1-p_1=k_2,\quad \ldots.\quad p_m+a_{m-1}-p_{m-1}=k_m;$$

and the coefficient in question is

$$\pm\Sigma\frac{a_{k_1+p_m-p_1}.a_{k_2+p_1-p_2}\ldots.a_{k_m+p_{m-1}-p_m}}{[p_1][k_2-p_1][p_2][k_3-p_2]\ldots[p_m][k_1-p_1]} \quad \ldots\ldots\ldots (a).$$

But by Professor CAYLEY'S method we may find a covariant of a function with binomial coefficients by operating with

$$(D_{x_1} D_{y_2} - D_{x_2} D_{y_1})^{k_1} (D_{x_2} D_{y_3} - D_{x_3} D_{y_2})^{k_2} \ldots (D_{x_m} D_{y_1} - D_{x_1} D_{y_m})^{k_m}.$$

If we divide out the factor $[k_1]\,[k_2] \ldots [k_m]$ which appears in the development, the coefficient of

$$\pm (D_{x_1} D_{y_2})^{k_1 - p_1} (D_{x_2} D_{y_1})^{p_1} \ldots\ldots (D_{x_m} D_{y_1})^{k_m - p_m} (D_{x_1} D_{y_m})^{p_m}$$

is $$\pm \frac{1}{[p_1]\,[k_1 - p_1]\,[p_2]\,[k_2 - p_2] \ldots [p_m]\,[k_m - p_m]}.$$

Also the corresponding argument in the leading term is

$$a_{k_1 - p_1 + p_m} \cdot a_{k_2 - p_2 + p_1} \cdot \ldots\ldots a_{k_m - p_m + p_{m-1}}.$$

This result agrees with (a). Hence the coefficient named in the question is the leading term of the covariant $(1.2)^{k_1} (2.3)^{k_2} \ldots (m.1)^{k_m}$, according to the Hyperdeterminant notation.

2647. (From WOLSTENHOLME'S *Book of Mathematical Problems.*)—In the equation $x^3 - qx + r = 0$ it is known that q and r both lie between $+1$ and -1; assuming all values between these limits to be equally probable, prove that the chance of all the roots of the equation being real is $\frac{2}{45}\sqrt{3}$.

Solution by J. J. WALKER, M.A.

The chance that q should have any one particular value between the assigned limits is $\frac{1}{2}dq$; that $(\frac{1}{3}r)^2$ should not be greater than $(\frac{1}{3}q)^3$, or r lie between $\pm 2\,(\frac{1}{3}q)^{\frac{3}{2}}$ is $4(\frac{1}{3}q)^{\frac{3}{2}} \div 2 = 2\,(\frac{1}{3}q)^{\frac{3}{2}}$. The chance then that q should be positive and $(\frac{1}{3}r)^2$ not greater than $(\frac{1}{3}q)^3$, the conditions necessary that the roots of $x^3 - qx + r = 0$ should be all real, is

$$\frac{1}{\sqrt{27}} \int_0^{\sqrt{1}} q^{\frac{3}{2}}\, dq = \frac{2}{5\sqrt{27}} = \frac{2\sqrt{3}}{45}.$$

2428. (Proposed by the Rev. J. BLISSARD.)—Prove that

(1) $$\ldots\ldots \frac{\Gamma(mx)}{\Gamma(nx)}\,(x=0) = \frac{n}{m};$$

(2) $$\ldots\ldots \frac{\Gamma(n)}{\Gamma(2n)} - \frac{\Gamma(n+2)}{\Gamma(2n+2)} \cdot \frac{\pi^2}{1.2} + \frac{\Gamma(n+4)}{\Gamma(2n+4)} \cdot \frac{\pi^4}{1.2.3.4} - \&c. = 0.$$

Solution by the PROPOSER.

1. Since $\log \Gamma(1+x) = -\gamma x + \frac{1}{2}S_2 x^2 - \&c.$, [See DE MORGAN'S *Differential and Integral Calculus*, p. 585,] it is evident that

$$\Gamma(1+x) = 1 + \text{terms in ascending powers of } x. \quad \text{Hence}$$

$$\frac{\Gamma(1+mx)}{\Gamma(1+nx)}\left(=\frac{mx\,\Gamma(mx)}{nx\,\Gamma(nx)} = \frac{m}{n}\cdot\frac{\Gamma(mx)}{\Gamma(nx)}\right) = 1 + \text{terms in ascending powers of } x,$$

therefore $\quad \frac{m}{n}\frac{\Gamma(mx)}{\Gamma(nx)}(x=0) = 1$ and $\frac{\Gamma(mx)}{\Gamma(nx)}(x=0) = \frac{n}{m}$.

2. In *Reprint*, Vol. III., p. 87, the following theorem is proved:—

$$fx - \frac{n}{m+n}f_1x\cdot\frac{x}{1} + \frac{n(n+1)}{(m+n)(m+n+1)}f_2x\cdot\frac{x^2}{1.2} - \&c.$$

$$= f0 + \frac{m}{m+n}f_10\cdot\frac{x}{1} + \frac{m(m+1)}{(m+n)(m+n+1)}f_20\cdot\frac{x^2}{1.2} + \&c. \ldots\ldots \text{(I.)}$$

Let $m=n$, and multiply both sides by $\frac{1.2\ldots(n-1)}{1.2\ldots(2n-1)}$, then

$$\frac{\Gamma(n)}{\Gamma(2n)}fx - \frac{\Gamma(n+1)}{\Gamma(2n+1)}\cdot f_1x\cdot\frac{x}{1} + \frac{\Gamma(n+2)}{\Gamma(2n+2)}\cdot f_2x\cdot\frac{x^2}{1.2} - \&c.$$

$$= \frac{\Gamma(n)}{\Gamma(2n)}f0 + \frac{\Gamma(n+1)}{\Gamma(2n+1)}\cdot f_10\cdot\frac{x}{1} + \frac{\Gamma(n+2)}{\Gamma(2n+2)}\cdot f_20\cdot\frac{x^2}{1.2} + \&c.$$

Now let $fx = \cos x$, then after slight reduction, we have

$$(1-\cos x)\left(\frac{\Gamma(n)}{\Gamma(2n)} - \frac{\Gamma(n+2)}{\Gamma(2n+2)}\cdot\frac{x^2}{1.2} + \frac{\Gamma(n+4)}{\Gamma(2n+4)}\cdot\frac{x^4}{1.2.3.4} - \&c.\right)$$

$$= \sin x\left(\frac{\Gamma(n+1)}{\Gamma(2n+1)}\cdot\frac{x}{1} - \frac{\Gamma(n+3)}{\Gamma(2n+3)}\cdot\frac{x^3}{1.2.3} + \&c.\right)$$

Let $x = \pi$, then $\frac{\Gamma(n)}{\Gamma(2n)} - \frac{\Gamma(n+2)}{\Gamma(2n+2)}\cdot\frac{\pi^2}{1.2} + \frac{\Gamma(n+4)}{\Gamma(2n+4)}\cdot\frac{\pi^4}{1.2.3.4} - \&c. = 0.$

Let $x = \frac{\pi}{2}$, then $\frac{\Gamma(n)}{\Gamma(2n)} - \frac{\Gamma(n+1)}{\Gamma(2n+1)}\cdot\frac{\pi}{2} - \frac{\Gamma(n+2)}{\Gamma(2n+2)}\cdot\frac{1}{1.2}\left(\frac{\pi}{2}\right)^2 + \&c. = 0.$

In (I) put $-n$ for n, and then put $m = n+1$, then we have

$$fx + \frac{n}{1}\cdot f_1x\cdot\frac{x}{1} + \frac{n(n-1)}{1.2}f_2x\cdot\frac{x^2}{1.2} + \&c.$$

$$= f0 + \frac{n+1}{1}f_10\cdot\frac{x}{1} + \frac{(n+1)(n+2)}{1.2}\cdot f_20\cdot\frac{x^2}{1.2} + \&c. \ldots\ldots \text{(II.)}$$

From this general formula, a great variety of novel and interesting results may (especially by the use of the Representative Notation) be readily obtained.

2568. (Proposed by Professor BALL.)—Express the roots of the equation $a^6x^6 + 18a^4(bd-c^2)x^4 + 81a^2(bd-c^2)^2x^2 + 27d^2(a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd) = 0$, in terms of α, β, γ the roots of the equation $ax^3 + 3bx^2 + 3cx + d = 0$.

Solution by J. J. WALKER, M.A.

The roots of the proposed equation are at once seen to be the differences of the reciprocals of those of the cubic.

2571. (Proposed by Professor BALL.)—The equation whose roots are the ratios of the roots of $ax^3+3bx^2+3cx+d=0$, is

$$a^2d^2x^6+(3a^2d^2-9abcd)\,x^5+(6a^2d^2-45abcd+27ac^3+27b^3d)\,x^4$$
$$+(7a^2d^2-54abcd+54ac^3-81b^2c^2+54b^3d)\,x^3$$
$$+(6a^2d^2-45abcd+27ac^3+27b^3d)\,x^2+(3a^2d^2-9abcd)\,x+a^2d^2=0.$$

Prove this, and show that its discriminant is

$$a^4d^4\,(ad-9bc)^2\,(b^2d-ac^2)^4\,(a^2d^2+4ac^3+4db^3-3b^2c^2-6abcd)^3;$$

and account *à priori* for the circumstance that, while in the equation b and c rise to the third power, a and d only reach the second.

Solution by J. J. WALKER, M.A.

The sum of the six ratios $\frac{\alpha}{\beta}, \frac{\alpha}{\gamma} \ldots \frac{\gamma}{\beta}$ is $-\frac{a}{d}\Sigma a^2\beta$ or $3\,\frac{-ad+3be}{ad}$.

The sum of the products taken two and two is

$$3+\Sigma\frac{\alpha}{\beta}+\frac{a}{d}\Sigma a^3+\frac{a^2}{d^2}\Sigma a^3\beta^3,$$

which becomes, after substituting for the symmetric functions their values in terms of the coefficients, and reducing,

$$\frac{3}{a^2d^2}\,(2a^2d^2-15abcd+9ac^3+9b^3d).$$

The sum of the products of the six ratios taken three and three is

$$2+2\Sigma\frac{\alpha}{\beta}+\Sigma\frac{\alpha^2}{\beta^2}=\frac{1}{a^2d^2}\,(-7a^2d^2+54abcd-54ac^3+81b^2c^2-54b^3d).$$

The transformed equation is therefore

$$a^2d^2(x^6+1)+3ad\,(ad-3bc)\,(x^5+x)+3\,(2a^2d^2-15abcd+9ac^2+9b^3d)\,(x^4+x^2)$$
$$+(7a^2d^2-54abcd+54ac^3-81b^2c^2+54b^3d)\,x^3=0.$$

The discriminant of the given equation is most readily calculated from the product of the squares of the differences of the roots. This is easily seen to be equal to

$$\frac{1}{a^{16}\beta^{16}\gamma^{16}}\,(\alpha^2-\beta^2)^2\,(\gamma^2-\alpha^2)^2\,(\beta^2-\gamma^2)^2\,(\alpha^2-\beta\gamma)^4\,(\beta^2-\alpha\gamma)^4\,(\gamma^2-\alpha\beta)^4\,(\alpha-\beta)^4$$
$$(\gamma-\alpha)^4\,(\beta-\gamma)^4,$$

or

$$\frac{1}{a^{16}\beta^{16}\gamma^{16}}\,(\alpha+\beta)^2\,(\gamma+\alpha)^2\,(\beta+\gamma)^2\,(\alpha^2-\beta\gamma)^4\,(\beta^2-\alpha\gamma)^4\,(\gamma^2-\alpha\beta)^4\,(\alpha-\beta)^6$$
$$(\gamma-\alpha)^6\,(\beta-\gamma)^6.$$

$$\Sigma\alpha\beta \qquad (\alpha+\beta)(\gamma+\delta)(\beta-\gamma) = \Sigma\alpha^2\beta - \alpha\beta\gamma = \frac{9bc-ad}{a^2},$$

$$\text{and} \qquad (\alpha^2-\beta\gamma)(\beta^2-\alpha\gamma)(\gamma^2-\alpha\beta) = \alpha\beta\gamma\cdot\Sigma\alpha^3 - \Sigma\alpha^2\beta^2 = \frac{27}{a^4}(b^3d-ac^3),$$

$$\text{also,} \quad (\alpha-\beta)^2(\gamma-\alpha)^2(\beta-\gamma)^2 = -\frac{27}{a^4}\left\{(ad-bc)^2 - 4(b^2-ac)(c^2-bd)\right\}.$$

Hence the required product of squares of differences is

$$-\frac{(27)^2}{a^4\Delta^2}(ad-9bc)^2\left\{(ad-bc)^2 - 4(b^2-ac)(c^2-bd)\right\}^2.$$

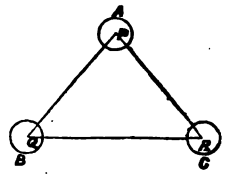
The circumstance of b and c rising to an index higher by 1 than the highest of a and d is common to all expressions which are the values of symmetric functions of the roots in terms of the coefficients, and into which those holding the same places from the two ends must enter similarly; and the reason of this is that each successive coefficient, divided by that of the highest power, is one degree higher relatively to the roots than its predecessor.

2523. (Proposed by T. SAVAGE, M.A.)—If A, B, C be three closed areas situated in such a manner that no straight line can be drawn meeting more than two of them, the average area of all triangles formed by joining three points taken at random in A, B, C respectively is equal to the area of the triangle formed by joining the centres of gravity of A, B, C.

2. If A, B, C, D be four closed volumes such that no plane can meet more than three of them, the average volume of the tetrahedron formed by joining four points taken at random in A, B, C, D respectively is equal to the volume of the tetrahedron formed by joining the centres of gravity of A, B, C, D respectively.

Solution by the REV. JOSEPH WOLSTENHOLME, M.A.

1. Take P, Q, R, three points in the areas A, B, C, and let du, dv, dw be elementary areas about P, Q, R. Then (Δ being the area of the triangle PQR) the average value of the triangle PQR



$$= \frac{1}{ABC}\iiint \Delta\, du\, dv\, dw;$$

but if z be the distance of R from PQ,

$$\int \Delta\, dw = \tfrac{1}{2}PQ\int z\, dw = \tfrac{1}{2}PQ\cdot\bar{z}\cdot C = C\cdot\Delta PQG_3,$$

G_1, G_2, G_3 being the centres of gravity of the three areas.

So $\int \Delta\, PQG_3\, dv = \tfrac{1}{2}PG_3\int y\, dv$, y being the distance of Q from PG_3,

$$= \tfrac{1}{2}PG_3\cdot\bar{y}\cdot B = \Delta\, PG_2G_3\cdot B,$$

and $\int \Delta\, PG_2G_3\, du = \tfrac{1}{2}G_2G_3\int x\, du = \tfrac{1}{2}G_2G_3\cdot\bar{x}\cdot A = \Delta\, G_1G_2G_3\cdot A$,

therefore the average value of Δ is the area of the triangle $G_1G_2G_3$.

It is necessary that all the area C shall be on the same side of PQ, otherwise the distance of G_3 from PQ, though the algebraical mean of the distances of points in C from that line, is not the average here desired. The areas must therefore be so placed that no straight line can be drawn intersecting all three.

2. Exactly the same method proves the corresponding property for four volumes, under like conditions.

2497. (Proposed by S. ROBERTS, M.A.)—Show that, if $a, b, c \ldots k$ denote $\frac{1}{1.2} \cdot \frac{d^2y}{dx^2}$, $\frac{1}{1.2.3} \cdot \frac{d^3y}{dx^3}$, $\frac{1}{1.2.3.4} \cdot \frac{d^4y}{dx^4}$, ... $\frac{1}{1.2 \ldots 9} \cdot \frac{d^9y}{dx^9}$, the general differential equation of

$$a_1x^3 + a_2x^2y + b_1xy^2 + b_2y^3 + a_3x^2 + dxy + b_3y^2 + c_1x + c_2y + c_3 = 0$$

is

$$\begin{vmatrix} a, b, c, a^2 & , & 0 & , & 0 \\ b, c, d, 2ab & , & a^2 & , & 0 \\ c, d, e, 2ac + b^2 & , & 2ab & , & a^3 \\ d, e, f, 2ad + 2bc & , & 2ac + b^2 & , & 3a^2b \\ e, f, g, 2ae + 2bd + c^2 & , & 2ad + 2bc & , & 3a^2c + 3ab^2 \\ f, g, h, 2af + 2be + 2cd, & & 2ae + 2bd + c^2, & & 3a^2d + 6abc + b^3 \end{vmatrix} = 0$$

Solution by the PROPOSER.

This is an exercise in determinants. We have to reduce

$$\begin{vmatrix} D^4y^3, & D^4xy^2, & D^4x^2y, & D^4y^2, & D^4xy, & D^4y \\ D^5y^3, & D^5xy^2, & D^5x^2y, & D^5y^2, & D^5xy, & D^5y \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ D^9y^3, & D^9xy^2, & D^9x^2y, & D^9y^2, & D^9xy, & D^9y \end{vmatrix} = 0,$$

where D means $\frac{d}{dx}$. Making use of $D^4xy^2 = xD^4y^2 + 4D^3y^2$, &c., we see that x does not appear explicitly in the result, and the determinant may be written

$$\begin{vmatrix} D^4y^3, & D^4y^2, & 4D^3y^2, & D^4y, & 4D^3y, & 6D^2y \\ D^5y^3, & D^5y^2, & 5D^4y^2, & D^5y, & 5D^4y, & 10D^3y \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ D^9y^3, & D^9y^2, & 9D^8y^2, & D^9y, & 9D^8y, & 36D^7y \end{vmatrix} = 0 \ldots\ldots\ldots (a).$$

To reduce this further, we write the first column in the form

$$\begin{aligned} &y^2D^4y + 4Dy^2\ D^3y + \ldots. + D^4y^2, \\ &y^2D^5y + 5Dy^2\ D^4y + \ldots. + D^5y^2, \\ &\quad \&c. \qquad\qquad \&c. \end{aligned}$$

Now, observing the forms of the remaining columns, we may write for the above

$$\left.\begin{array}{l} 0 \\ 10D^3y^2D^2y \\ 20D^3y^2D^3y + 15D^4y^2D^2y \\ 35D^3y^2D^4y + 35D^4y^2D^3y + 21D^5y^2D^2y \\ 56D^3y^2D^5y + 70D^4y^2D^4y + 56D^5y^2D^3y + 28D^6y^2D^2y \\ 84D^3y^2D^6y + 126D^4y^2D^5y + 126D^5y^2D^4y + 84D^6y^2D^3y + 36D^7y^2D^2y \end{array}\right\} \ldots(1).$$

Again, we may substitute $2yB + 6y'A$ for D^3y^2, $2yC + 8y'B + 6A^2$ for D^4y^2, $2yD + 10y'E + 20ab$ for $5Dy^2$, &c., where y', A, B, C, D, &c., are written for Dy, D^2y, D^3y, D^4y, D^5y, &c. These substitutions being made in the second and third columns, it is easily seen that for those columns we may write

$$\left.\begin{array}{l} 6A^2 \\ 20AB \\ 30AC + 20B^2 \\ 42AD + 70BC \\ 56AE + 112BD + 70C^2 \\ 72AF + 168BE + 252CD \end{array}\right\} (2), \qquad \left.\begin{array}{l} 0 \\ 5.\ 6A^2 \\ 6.20AB \\ 7.30AC + 20B^2 \\ 8.42AD + 70BC \\ 9.56AE + 112BD + 70C^2 \end{array}\right\} (3).$$

Making the same substitutions in (1), and comparing the result with (2) and (3), we find that for (1) may be written

$$\left.\begin{array}{l} 0 \\ 0 \\ 15A^3 \\ 105A^2B \\ 210A^2C + 28AB^2 \\ 378A^2D + 1260ABC + 280B^3 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(1').$$

We have only to replace the three first columns of (a) by (1'), (2). (3), and then put $1.2a$ for A, $1.2.3b$ for B, &c. The irrelevant numerical factors being divided out, the determinant takes the form given in the question.

It will be observed that the terms of the determinant are derivable from one another in a symmetrical manner; and the result, together with the corresponding differential equations for functions of the first and second degrees (Boole, quoting Monge, *Differential Equations*, p. 20), indicates a general form by analogy.

2461. (Proposed by S. ROBERTS, M.A.)—If a curve of the nth degree be translated in one direction, the centre of mean distances of the points common to any two positions of the curve lies always on a straight line.

Solution by F. D. THOMSON, M.A.

Take the axis of x in the direction of translation, and let the equation to the curve in its *original* position be $F(x, y) = 0$ .....(1);

and therefore in the *final* position the equation will be of the form

$$F(x+h, y) = 0, \text{ or } F(x,y) + h\frac{dF}{dx} + \frac{h^2}{\lfloor 2}\frac{d^2F}{dx^2} + \ldots + \frac{h^n}{\lfloor n}\frac{d^nF}{dx^n} F = 0 \ldots (2).$$

Hence, combining (1) and (2), we have for the points of intersection of the two curves

$$F(x,y) = 0, \text{ and } \frac{dF}{dx} + \frac{h}{\lfloor 2}\frac{d^2F}{dx^2} + \frac{h^2}{\lfloor 3}\frac{d^3F}{dx^3} + \ldots + \frac{h^{n-1}}{\lfloor n}\frac{d^nF}{dx^n} = 0 \ldots . (1, 3.)$$

We shall have to eliminate x between (1) and (3).

Now let $\frac{x}{y} = u$, and substitute for x in (1) and (3).

Then $(1) \equiv y^n f_n(u) + y^{n-1} f_{n-1}(u) + \ldots + y f_1(u) + C = 0$

(the *suffixes* serving to denote the highest power of u in the coefficients of the powers of y),

and $$(2) \equiv \frac{1}{y}\frac{dF}{du} + \frac{h}{\lfloor 2}\frac{1}{y^2}\frac{d^2F}{du^2} + \ldots + \frac{h^{n-1}}{\lfloor n}\frac{1}{y^n}\frac{d^nF}{du^n} = 0.$$

Now
$$\frac{dF}{du} = y^n f'_n(u) + y^{n-1} f'_{n-1}(u) + \ldots + y f_1'(u),$$
$$\frac{d^2F}{du^2} = y^n f''_n(u) + y^{n-1} f''_{n-1}(u) + \ldots + y^2 f_2''(u),$$
$$\&c. = \&c.,$$
$$\frac{d^nF}{du^n} = y^n f^n_n(u);$$

therefore
$$\frac{1}{y}\frac{dF}{du} = y^{n-1} f'_n(u) + y^{n-2} f'_{n-1}(u) + \ldots + f_1'(u),$$
$$\frac{h}{\lfloor 2}\cdot\frac{1}{y^2}\frac{d^2F}{du^2} = \frac{h}{2} y^{n-2} f''_n(u) + \ldots + \frac{h}{2} f_2''(u),$$
$$\&c. = \&c.,$$
$$\frac{h^{n-1}}{\lfloor n}\cdot\frac{1}{y^n}\frac{d^nF}{du^n} = \frac{h^{n-1}}{\lfloor n} f^n_n(u);$$

therefore
$$(2) \equiv y^{n-1} f'_n(u) + y^{n-2}\left\{f'_{n-1}(u) + \frac{h}{2} f''_n(u)\right\} + \&c.$$
$$+ \left\{f_1'(u) + \frac{h}{2} f_2''(u)\right\} + \&c. = 0.$$

Now all the n values of u from (1) may be shown to be given in terms of y by the expansion $u = \alpha + \frac{\beta}{y} + \frac{\gamma}{y^2} + \&c.$

Where α is one of the roots of $f_n(\alpha) = 0$,

$$\beta = -\frac{f_{n-1}(\alpha)}{f'_n(\alpha)}, \ \&c.,$$

provided that the equation $f_n(\alpha) = 0$ has no *equal* roots.

(2), we get

$$\ldots\Big\}$$

$$\ldots + \frac{\beta}{\gamma} f''_{n-1}(a) + \frac{h}{2} f''_n(a) + \ldots \Big\} + \&c. = 0,$$

$$\ldots f''_n(a) + f'_{n-1}(a) + \frac{h}{2} f''_n(a) \Big\} + \&c. = 0;$$

$$\ldots f''_n(a) + f'_{n-1}(a) + \frac{h}{2} f''_n(a)$$

$$\ldots \frac{}{f'_n(a)}$$

... right-hand side denotes that all the roots of the equation ... substituted successively for a, and the results summed.

$$\Sigma y = \text{constant} - \frac{h}{2} \Sigma \frac{f''_n(a)}{f'_n(a)}.$$

$$\ldots (x-\alpha)(x-\beta)(x-\gamma)\ldots(x-\lambda),$$

$$\ldots \frac{1}{x-\alpha} + \frac{1}{x-\beta} + \frac{1}{x-\gamma} + \frac{1}{x-\delta} + \ldots + \frac{1}{x-\lambda},$$

$$\ldots = 0, \qquad \text{therefore} \qquad \Sigma y = \text{constant},$$

... position of the points of intersection moves along ... direction of translation.

... theorem does not hold good if the terms of highest ...

... WORTH.)—1. The number of ways ... allotted to r different persons, blank ...

... number of ways is the coefficient of ...

... alike can be allotted to r ... the number of combinations ...

... ways is the number of ...

... N.A.

... of the r persons, ...

2. From the last number must be subtracted the number of ways in which A, or B, or C, ... may be void, that is $r(r-1)^n$. In performing this subtraction we have subtracted twice all the ways in which two, as A, B, may be void, once in the portion subtracted for A and once for B. We must therefore add this number, which is $\frac{1}{2}r(r-1)(r-2)^n$. We have now subtracted the number of ways in which A, B, C may be void three times, and added it three times (in BC, CA, AB); we must therefore subtract it, and so on, exactly as in finding the number of numbers less than N and prime to it. The whole number of ways in which the n things may be distributed so that every body has at least one is then

$$r^n - r(r-1)^n + \frac{r(r-1)}{\lfloor 2}(r-2)^n - \ldots = \text{coefficient of } x^n \text{ in } (\epsilon^x - 1)^r \lfloor n.$$

[See Wolstenholme's *Book of Mathematical Problems*, Exs. 163 and 239. Mr. Wolstenholme remarks that he gave these results to Mr. C. B. Clarke, to set in Queen's College, Cambridge, in either 1865 or 1866, before the appearance of Mr. Todhunter's *History of the Theory of Probability*, in which he discovered they had been known more than a century.]

3 and 4. The result in (3) is clearly the number of solutions in integers of the equation $x_1 + x_2 + \ldots + x_r = n$, zero allowed, and is therefore the coefficient of x^n in the expansion of $(1 + x + x^2 + \ldots)(1 + x + x^2 + \ldots) \ldots$ to r factors; or in the expansion of $(1-x)^{-r}$. So if zero values be not allowed, the result required is the coefficient of x^n in $x^r(1-x)^{-r}$.

These results are $\dfrac{\lfloor r+n-1}{\lfloor r-1 \,.\, \lfloor n}$, and $\dfrac{\lfloor n-1}{\lfloor r-1 \,.\, \lfloor n-r}$.

[The PROPOSER remarks that Professor CAYLEY'S theorem (Quest. 2309) follows immediately as a corollary from (2). He also gives the following proof of the fourth part of the question:—

4. Suppose the n things all alike be placed in a row; we may then place amongst them $r-1$ points of partition and assign the r parts in order to the r different persons in order. The number of ways in which it can be done is therefore the number of ways in which $r-1$ points of partition may be assigned out of $n-1$ intervals. It is therefore the number of combinations of $(n-1)$ things $(r-1)$ at a time.]

2591. (Proposed by Professor WHITWORTH.) — Let AA_1, BB_1, CC_1 be the bisectors of the triangle ABC, intersecting in G_1; let AA_2, BB_2, CC_2 be the perpendiculars from the angular points on the opposite sides, intersecting in G_2; and let B_1C_1, B_2C_2 intersect in P; C_1A_1, C_2A_2 in Q; and A_1B_1, A_2B_2 in R. Then AP, BQ, CR will all be perpendicular to G_1G_2; and the triangle PQR will "circumscribe" the triangle ABC. Let A_i, B_i, C_i be the feet of the interior bisectors of the angles A, B, C, and A_e, B_e, C_e the feet of the exterior bisectors: then the six straight lines PA_i, QB_i, RC_i, PA_e, QB_e, RC_e pass three and three through four points which are the points of contact of the nine-point circle with the inscribed and escribed circles.

Solution by STEPHEN WATSON.

The equations of the bisectors and perpendiculars being comprehended in

$$\sin A \cdot \alpha = \sin B \cdot \beta = \sin C \cdot \gamma, \quad \cos A \cdot \alpha = \cos B \cdot \beta = \cos C \cdot \gamma,$$

the equation of the line G_1G_2 is easily found to be

$$\sin 2A \sin (B-C) \cdot \alpha + \sin 2B \sin (C-A) \cdot \beta + \sin 2C \sin (A-B) \cdot \gamma = 0 \ldots (1);$$

also the points P, Q, R are determined by the equations

$$-\frac{\alpha}{\sin (B-C)} = \frac{\beta}{\sin (C-A)} = \frac{\gamma}{\sin (A-B)} \quad \ldots\ldots\ldots\ldots (2),$$

$$\frac{\alpha}{\sin (B-C)} = -\frac{\beta}{\sin (C-A)} = \frac{\gamma}{\sin (A-B)} \quad \ldots\ldots\ldots\ldots (3),$$

$$\frac{\alpha}{\sin (B-C)} = \frac{\beta}{\sin (C-A)} = -\frac{\gamma}{\sin (A-B)} \quad \ldots\ldots\ldots\ldots (4).$$

For the equations of B_1C_1, B_2C_2 are

$$\sin A \cdot \alpha - \sin B \cdot \beta - \sin C \cdot \gamma = 0, \quad \cos A \cdot \alpha - \cos B \cdot \beta - \cos C \cdot \gamma = 0,$$

from which eliminating successively α, β, γ, we get the equalities in (2), and similarly (3) and (4) may be proved.

Take (1); first with the equation of AP in (2); next with that of BQ in (3); and lastly with that of CR in (4); in each case, it is easily shown, the lines are perpendicular; hence AP, BQ, CR are all perpendicular to G_1G_2.

From (2), (3), (4) it is plain each of the triads of points A, Q, R; B, R, P; C, P, Q lie in a straight line; hence the triangle PQR circumscribes the triangle ABC.

Again, from (2), (3), (4) and the well-known equations of the interior and exterior bisectors of the angles, we find the following equations:

$$PA_s \ldots\ldots \sin \tfrac{1}{2}(B+C) \cdot \alpha - \sin \tfrac{1}{2}(B-C)(\beta-\gamma),$$

$$QB_s \ldots\ldots \sin \tfrac{1}{2}(C+A) \cdot \beta - \sin \tfrac{1}{2}(C-A)(\gamma-\alpha),$$

$$RC_s \ldots\ldots \sin \tfrac{1}{2}(A+B) \cdot \gamma - \sin \tfrac{1}{2}(A-B)(\alpha-\beta),$$

$$PA_e \ldots\ldots \cos \tfrac{1}{2}(B+C) \cdot \alpha + \cos \tfrac{1}{2}(B-C)(\beta+\gamma),$$

$$QB_e \ldots\ldots \cos \tfrac{1}{2}(C+A) \cdot \beta + \cos \tfrac{1}{2}(C-A)(\gamma+\alpha),$$

$$RC_e \ldots\ldots \cos \tfrac{1}{2}(A+B) \cdot \gamma + \cos \tfrac{1}{2}(A-B)(\alpha+\beta),$$

and from these equations we find that the triads of lines

$$(PA_s, QB_s, RC_s), (PA_s, QB_e, RC_e), (QB_s, RC_e, PA_e), (RC_s, PA_e, QB_e)$$

pass respectively through the points determined by

$$\alpha : \beta : \gamma = \sin^2 \tfrac{1}{2}(B-C) : \sin^2 \tfrac{1}{2}(C-A) : \sin^2 \tfrac{1}{2}(A-B) \ldots (5),$$

$$\alpha : \beta : \gamma = -\sin^2 \tfrac{1}{2}(B-C) : \cos^2 \tfrac{1}{2}(C-A) : \cos^2 \tfrac{1}{2}(A-B) \ldots (6),$$

$$\alpha : \beta : \gamma = \cos^2 \tfrac{1}{2}(B-C) : -\sin^2 \tfrac{1}{2}(C-A) : \cos^2 \tfrac{1}{2}(A-B) \ldots (7),$$

$$\alpha : \beta : \gamma = \cos^2 \tfrac{1}{2}(B-C) : \cos^2 \tfrac{1}{2}(C-A) : -\sin^2 \tfrac{1}{2}(A-B) \ldots (8).$$

Now, in the *Messenger of Mathematics*, Vol. IV., p. 63, it is shown that (5) is the point of contact of the inscribed and nine-point circles, and in a similar manner it may be shown that (6), (7), (8) are the points of contact of the escribed circles with the same circle.

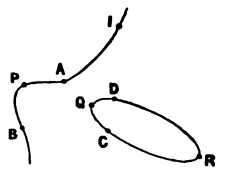
NOTE ON PRIME NUMBERS: by A. DE MORGAN.

The following matter is slight, but curious, as showing the possibility of proving a proposition in the theory of numbers by a method which extends only to an assignable integer, and then fails.

It is said, but I believe not proved, that an even number is always the sum of two prime numbers. This, I suppose, was suggested by trial. Now if the proposition be *false* in the case of $2n$, we must have $2n$ — (any less odd prime) = (some less odd composite); and different primes require different composites. Therefore the number of less odd composites must be at least equal to the number of less odd primes. But this does not happen until $2n = 120$; therefore, so far, the theorem must be true.

2697. (Proposed by Professor SYLVESTER.)—Let I be a point of inflexion on a cubic. From I draw tangents meeting the cubic in P, Q, R; and from P draw tangents meeting it in A, B, C, D. Prove that the three opposite pairs of sides of the tetragon ABCD will respectively intersect in I, Q, R.

Solution by F. D. THOMSON, M.A.



I is the point of inflexion, P, Q, R the points of contact of the three tangents from it, A, B, C, D the points of contact of the four tangents from P.

Then since III is a straight line, the satellites PQR are in a straight line.

Again, since PAA, and also PBB, are in a straight line, therefore the satellites of PP, AB, AB are in a straight line. But I is the satellite of PP or P; therefore if x be the satellite of AB, I is the satellite of x. But the only points of which I is the satellite are I, P, Q, R; therefore x must coincide with I, Q, or R.

Similarly, each of the six chords joining A, B, C, D passes through either I, Q, or R. If, for example, AB passes through I, BC through R, and therefore BD through Q, then the satellites of AC, BB, IR are in a straight line; but PQR is a straight line, hence AC passes through Q.

Hence BD, CA meet in Q; AD, BC in R; and BA, CD in I.

2694. (Proposed by G. O. HANLON.)—A rope without weight passes over a smooth pulley, having a monkey at one end and a weight equal to the monkey at the other. The monkey suddenly runs up the rope to a height h with a uniform velocity; what is the resulting motion of the weight?

Solution by SEPTIMUS TEBAY, B.A.

Let P be the mass of the monkey, Q the mass of the weight at the other end of the rope, x the space through which Q has ascended at the time t, and T the tension of the rope. Then if the monkey begins to ascend with a velocity a, its position at the time t will be $at - x$.

Now the accelerating forces on P, Q are $\frac{T-Pg}{P}$, $\frac{T-Qg}{Q}$.

Therefore $\frac{d^2}{dt^2}(at-x) = \frac{T-Pg}{P}$, $\frac{d^2x}{dt^2} = \frac{T-Qg}{Q}$;

and eliminating T, $Q\frac{d^2x}{dt^2} - P\frac{d^2}{dt^2}(at-x) = (P-Q)g$.

Therefore $Q\frac{dx}{dt} - P\frac{d}{dt}(at-x) = (P-Q)gt + C$,

or $(P+Q)\frac{dx}{dt} = (P-Q)gt + Pa + C$.

To determine C, we must conceive the weight P to receive a sudden impulse Pa, which will produce an initial velocity in $Q = \frac{Pa}{P+Q}$. Therefore $C = 0$; and $x = \frac{P-Q}{P+Q}\cdot\frac{g}{2}t^2 + \frac{Pat}{P+Q}$.

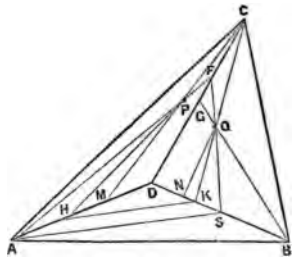
When $P = Q$, $x = \frac{1}{2}at$. Also, $h = at - x$.

2614. (Proposed by Capt. CLARKE, R.E., F.R.S.)— Three points are taken at random, one on each of three faces of a tetrahedron, show that the chance that the plane passing through them cuts the fourth face is $\frac{3}{4}$.

Solution by the PROPOSER.

Let D be the vertex of the tetrahedron, P, Q any points in the faces ADC, BDC. Produce AP, BQ to F and G, and suppose DG less than DF. Produce CP, CQ to meet AD, BD in H and K; let FQ produced meet DB in S; and join AS. Draw PM, QN parallel to CD. If the third point R be taken within the quadrilateral HASK, it is clear that the plane PQR will not cut the face ABC. Now the chance that R falls in this space is

$$\frac{AD.DS}{AD.DB} - \frac{HD.DK}{AD.DB} \quad \ldots\ldots (\alpha).$$



Let $DA = f$, $DB = g$, $DC = h$, $DM = fx$, $MP = hy$, $DN = gx'$, $NQ = hy'$, $DF = nh$; $HD = \mu f$, $DK = \nu g$, $DS = \lambda g$; then we have

$$x + \frac{y}{n} = 1, \quad \frac{x}{\mu} + y = 1, \quad \frac{x'}{\nu} + y' = 1, \quad \frac{x'}{\lambda} + \frac{y'}{n} = 1;$$

therefore $n = \frac{y}{1-x}$, $\lambda = \frac{x'}{1-\frac{y'}{n}}$, $\mu = \frac{x}{1-y}$, $\nu = \frac{x'}{1-y'}$.

Substituting in (a), it becomes $\dfrac{x}{1-\frac{y'}{n}} - \dfrac{xx'}{(1-y)(1-y')}$.

The equation to the line FB is $x' + \frac{y'}{n} = 1$, so that, if we integrate with respect to x', y', x, y, successively, the probability that the plane PQR does not cut ABC is

$$4\int_0^1\int_0^{1-y}\int_0^n\int_0^{1-\frac{y'}{n}}\left[\frac{nx'}{n-y'} - \frac{xx'}{(1-y)(1-y')}\right]dy\,dx\,dy'\,dx'$$

$$= 2\int_0^1\int_0^{1-y}\int_0^n\left[1 - \frac{1-x}{y}y' - \frac{x}{1-y}\cdot\frac{\left(1-\frac{1-x}{y}y'\right)^2}{1-y'}\right]dy\,dx\,dy'$$

$$= \int_0^1\int_0^{1-y}\left[\frac{y}{1-x} + 2\frac{x(1-x)}{y(1-y)} - \frac{3x}{1-y} + \frac{2x}{1-y}\cdot\frac{(1-x-y)^2}{y^2}\log\frac{1-x-y}{1-x}\right] \times dy\,dx.$$

Let F, G, H, K be the values of the four separate terms in this expression,

then $F = \frac{1}{4}$; $G = 2\int_0^1\left[\frac{1}{6} + \frac{1}{6y} - \frac{1}{3}y\right]dy = \frac{1}{3}\left[\log y\right]_0^1$; $H = -\frac{3}{4}$;

$$K = 2\int_0^1\int_0^{1-y}\frac{1}{y^2}\left[x(1-y) - 2x^2 + \frac{x^3}{1-y}\right]\log\frac{1-x-y}{1-x}\,dy\,dx$$

$$= -\frac{1}{6}\int_0^1\left[\left(y + \frac{3}{1-y} - 3\right)\log y + \left(\frac{1}{y^2} - \frac{3}{y} + 3 - y\right)\log(1-y) + \frac{9}{2} - \frac{1}{y} - \frac{y}{2}\right]dy$$

$$= \left[-\frac{1}{2}\log y\log(1-y) + \left(\frac{1-y^2}{12} - \frac{1-y}{2}\right)\log(1-y) + \left(\frac{y^2}{12} - \frac{y}{2}\right)\log y\right.$$
$$\left. + \frac{1}{12}y - \frac{1-y}{6y}\log(1-y) - \frac{1}{3}\log y + \frac{3}{4}y - \frac{1}{24}y^2\ldots\right]_0^1 = \frac{5}{8} - \frac{1}{3}\left[\log y\right]_0^1;$$

therefore $F + G + H + K = \frac{1}{8}$.

This result is obtained subject to the condition that DG is less than DF. It is clear that, for the case in which DG is greater than DF, we should obtain also $\frac{1}{8}$. Therefore the chance that the plane through PQR does not cut the base is $\frac{1}{4}$; that it *does*, the chance is $\frac{3}{4}$.

2604. (Proposed by J. J. WALKER, M.A.)—Prove that in the third system of pulleys (these being supposed n in number, of equal weight w, and equal diameter $2a$), the distance of the centre of the parallel tensions of the cords from the first cord is equal to

$$\frac{4\left\{(n-1)2^{n-1}+1\right\}W + n\left\{(n-3)2^n + n + 3\right\}w}{2(2^n-1)W}\,a.$$

Solution by the PROPOSER.

If P is the power which, applied to the free end of the first cord, will just support the weight W, and $T_1, T_2, \ldots T_n$ represent the tensions of the successive cords, we have

$$P = T_1,\ 2T_1 = T_2 - w,\ 2T_2 = T_3 - w, \ldots . 2T_{n-1} = T_n - w,\ T_1 + T_2 \ldots + T_n = W.$$

Multiplying these equations by $2^n - 1, 2^{n-1} - 1, \ldots . 2 - 1, 1$, respectively, and adding all together, $T_1, T_2, \ldots . T_n$ are eliminated, and we have

$$(2^n - 1)P = W - (2^n - 1 - n)w, \text{ or } P = \frac{W - (2^n - 1 - n)w}{2^n - 1}.$$

$$\text{Again } T_1 = P,\ T_2 = 2T_1 + w = 2P + w = \frac{2W - (2^n - 1 - 2n)w}{2^n - 1},$$

$$T = 2T_2 + w = \frac{2^2 W - (2^n - 1 - 2^2 n)w}{2^n - 1} \ldots .$$

$$T_n = 2T_{n-1} + w = \frac{2^{n-1} W - (2^n - 1 - 2^{n-1} n)w}{2^n - 1}.$$

Now, if x be the required distance,

$$\frac{xW}{a} = T_2 + 2T_3 + 3T_4 + \ldots . + (n-1)T_n,$$

$$\text{or } \frac{x(2^n - 1)W}{a} = (1.2 + 2.2^2 + 3.2^3 + \ldots . + \overline{n-1}\,2^{n-1})(W + nw)$$

$$- (1 + 2 + 3 \ldots . + n - 1)(2^n - 1)w$$

$$= \left\{(n-1)2^n + 2\right\}(W + nw) - \tfrac{1}{2}n(n-1)(2^n - 1)w$$

$$= 2\left\{(n-1)2^{n-1} + 1\right\}W + \tfrac{1}{2}\left\{(n-3)2^n + n + 3\right\}nw.$$

2486. (Proposed by G. O. HANLON.)—Find a point on an ellipse such that the normal produced to meet the curve again may cut off a maximum area.

Solution by M. JENKINS, B.A.; *the* PROPOSER; *and others.*

By taking consecutive normals we see that the chord formed by the normal produced must be bisected at its stationary point. Hence the osculating circle must cut the chord again in the same point as the ellipse. Therefore the chord, being a common chord of the ellipse and osculating circle, is equally inclined to the axis with the tangent (corresponding to the normal).

Hence the tangent and normal at the required point must be equally inclined to the axis; and we can thus easily show that the tangent of the inclination to the major axis of the radius vector to the required point is equal to the square of the ratio of the minor to the major axis.

Let CA, CB be the semi-major and semi-minor axes respectively of the ellipse; CU a radius vector of the auxiliary circle such that $\tan \mathrm{CUA} = \frac{\mathrm{BC}}{\mathrm{AC}}$. Then, if UPN be an ordinate cutting the ellipse in P, P is one of the required points.

We shall also have the following properties. Let POH be the normal chord; O the middle and stationary point of the chord. Then POH is the tangent at O to the evolute of the ellipse. But CO and CP are conjugate respectively to the normal and tangent at P. Hence CO and CP are also equally inclined to the axes; and therefore CP cuts the evolute in a point such that the tangent at it is conjugate to CP and cuts off a maximum area.

[Under a slightly different form of enunciation, the problem is to find in an ellipse a point such that, if there be drawn through it a diameter and a normal chord, the area contained by these lines and the curve shall be a maximum. Putting a and b for the semi-axes, $c^2 = a^2 - b^2$, and θ for the inclination of the required normal to the major axis, the area in question is

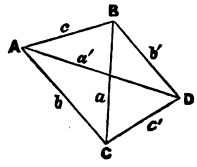
$$\frac{2a^2b^2c^2 \sin 2\theta}{c^4 \sin^2 2\theta + 4a^2b^2} + ab \tan^{-1}\left(\frac{c^2 \sin 2\theta}{2ab}\right),$$

which is readily found to have its maximum value $\frac{2a^2b^2c^2}{(a^2+b^2)^2} + ab \tan^{-1}\left(\frac{c^2}{2ab}\right)$ when $\theta = \frac{1}{4}\pi$; the position of the normal thus determined being the same as that found in Mr. JENKINS'S solution.]

2690. (Proposed by M. COLLINS, B.A.)—If four points on a plane be the extremities of the two straight lines a, a', or of the pair b, b', or of the pair c, c'; then, a, b, c being the sides of a triangle, prove that

$$(a^2a'^2 + b^2b'^2 + c^2c'^2)(a^2 + a'^2 + b^2 + b'^2 + c^2 + c'^2) = 2(a^4a'^2 + a'^4a^2 + b^4b'^2 + b'^4b^2 + c^4c'^2 + c'^4c^2) + a^2b^2c^2 + a^2b'^2c'^2 + a'^2b^2c'^2 + a'^2b'^2c^2.$$

Solution by S. BILLS; J. WILSON; *and others.*



Let A, B, C, D be the four points; then, since the angle ACD = ACB + DCB, we shall have the well-known relation

$$1 - \cos^2 \mathrm{ACB} - \cos^2 \mathrm{DCB} - \cos^2 \mathrm{ACD} + 2 \cos \mathrm{ACB} \cos \mathrm{DCB} \cos \mathrm{ACD} = 0 \ldots (1).$$

But $\cos \mathrm{ACB} = \frac{a^2 + b^2 - c^2}{2ab}$, $\cos \mathrm{DCB} = \frac{a^2 + c'^2 - b'^2}{2ac'}$,

and $\cos \mathrm{ACD} = \frac{b^2 + c'^2 - a'^2}{2bc'}$.

Substituting these results in (1), we readily find

$$4a^2b^2c'^2 - c'^2(a^2 + b^2 - c^2)^2 - b^2(a^2 + c'^2 - b'^2)^2 - a^2(b^2 + c'^2 - a'^2)^2 + (a^2 + b^2 - c^2)(a^2 + c'^2 - b'^2)(b^2 + c'^2 - a'^2) = 0 \ldots . (2).$$

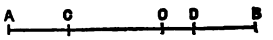
This equation (2) is equivalent to the one in the question, and may be easily reduced to the same form.

2695. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—If a given finite straight line of length a be divided at random in two points, the chance that the product of the three parts shall exceed $\frac{1}{108}a^3$ is

$$\frac{4}{9}\int_{\frac{1}{9}\pi}^{\frac{8}{9}\pi} \sin\tfrac{1}{2}\theta\,(1+2\cos 3\theta)^{\frac{1}{2}}\,d\theta.$$

Solution by STEPHEN WATSON.

Let C and D be the points in the given line AB, and bisect GB in O. Put $AC=ax$, $OD=ay$; then by the question



$$x\left\{\tfrac{1}{4}(1-x)^2-y^2\right\} > \frac{1}{108}, \quad \text{therefore} \quad y < \pm\tfrac{1}{2}\left\{(1-x)^2-\frac{1}{27x}\right\}^{\frac{1}{2}};$$

hence the number of values y takes while x remains fixed is $\left\{(1-x)^2-\frac{1}{27x}\right\}^{\frac{1}{2}}$; therefore, doubling because C and D may be interchanged, the required chance is

$$2\int\left\{(1-x)^2-\frac{1}{27x}\right\}^{\frac{1}{2}}dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

the limits being derived from $(1-x)^2x=\frac{1}{27}$, in which, if we put $\frac{2}{3}(1+\cos\theta)$ for x, we have $4\cos^3\theta-3\cos\theta=-\frac{1}{2}$, or $\cos 3\theta=-\frac{1}{2}$, whence $\theta=\frac{2}{9}\pi$, or $\frac{4}{9}\pi$, or $\frac{8}{9}\pi$. The first of these values of θ gives $x>1$, and is therefore inadmissible; the second gives the greater value of x, and the third the less. Putting, then, $x=\frac{2}{3}(1+\cos\theta)$ in (1), it easily gives the probability stated in the question.

2687. (Proposed by W. S. M'CAY, B.A.)—If normals to circumscribing conics at the vertices of a triangle meet in a point, and if also normals to inscribed conics at the points of contact meet in a point; show that the locus of this point is the same in both cases.

Solution by STEPHEN WATSON.

If the equation of the circumscribing conic be $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$, the tangents at the angles are

$$n\beta+m\gamma=0, \quad l\gamma+n\alpha=0, \quad \text{and} \quad m\alpha+l\beta=0;$$

and the normals, being respectively perpendicular to those, are easily found to be

$$\left.\begin{aligned} n\,(\gamma+\beta\cos A) &= m\,(\beta+\gamma\cos A)\\ l\,(\alpha+\gamma\cos B) &= n\,(\gamma+\alpha\cos B)\\ m\,(\beta+\alpha\cos C) &= l\,(\alpha+\beta\cos C)\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (1).$$

Again, let the inscribed conic be $(l'\alpha)^{\frac{1}{2}}+(m'\beta)^{\frac{1}{2}}+(n'\gamma)^{\frac{1}{2}}=0$, then the normals at the points of contact are

$$\left.\begin{aligned} n'(\gamma + \alpha\cos B) &= m'(\beta + \alpha\cos C) \\ l'(\alpha + \beta\cos C) &= n'(\gamma + \beta\cos A) \\ m'(\beta + \gamma\cos A) &= l'(\alpha + \gamma\cos B) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (2):$$

for the first must be parallel to $\gamma + \alpha\cos B = 0$, and pass through the point $(\alpha = 0,\ m'\beta = n'\gamma)$, hence its equation is found to be as above. Similarly for the others.

Eliminating (l, m, n) from (1), and (l', m', n') from (2), we obtain in both cases the same result as the equation of the required locus, viz.,

$$(\cos A - \cos B\cos C)(\beta^2 - \gamma^2)\alpha + (\cos B - \cos C\cos A)(\gamma^2 - \alpha^2)\beta + (\cos C - \cos A\cos B)(\alpha^2 - \beta^2)\gamma = 0 \quad \ldots\ldots (3).$$

Put successively $\alpha, \beta, \gamma = 0$; then the results give

$$\frac{\alpha}{\cos A - \cos B\cos C} = \frac{\beta}{\cos B - \cos C\cos A} = \frac{\gamma}{\cos C - \cos A\cos B} \quad \ldots (4);$$

hence (3) meets the sides of the triangle of reference in three points such that, when joined to the opposite angles, the three joining lines are concurrent.

2627. (Generalized from WHITWORTH'S *Choice and Chance*, page 65.)— How many different signals can be made with a set of n different flags, using r of them at a time, and arranging them on s masts where $n > r > s$?

Solution by PROFESSOR WHITWORTH.

The r flags can be selected and arranged in one row in $\frac{\lfloor n}{\lfloor n-r} \div \lfloor r$ different ways. And then $s-1$ points of partition can be placed amongst them in $\frac{\lfloor r-1}{\lfloor r-s \lfloor s-1}$ different ways. Hence the number of signals will be

$$\frac{\lfloor n \lfloor r-1}{\lfloor n-r \lfloor r-s \lfloor s-1}.$$

If it be permitted to leave some of the masts vacant, then by Question 2414 (3) the $s-1$ points of partition can be placed in $\frac{\lfloor s+r-1}{\lfloor r \lfloor s-1}$ different ways and the number of signals will therefore be

$$\frac{\lfloor n \lfloor s+r-1}{\lfloor n-r \lfloor r \lfloor s-1}.$$

2506. (Proposed by the Rev. R. TOWNSEND, F.R.S.) — Determine the two positions of the line whose chord of intersection with an assigned one of three given circles shall be cut harmonically by the other two.

I. *Solution by the* PROPOSER.

Let A, B, C be the three circles, and P, Q, R their three centres; then, since a chord XY of C cut harmonically by A and B, being the diameter of a circle orthogonal to A and B, must pass through a determinable point O on the line PQ and have its middle point Z on the radical axis L of A and B, therefore the circle on OR as diameter will intersect L at the two middle points Z_1 and Z_2 of the two required chords X_1Y_1 and X_2Y_2 through O, and therefore, &c.

II. *Solution by the* REV. J. WOLSTENHOLME, M.A.

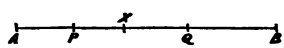
Let the assigned circle be A, the two others B, C. A straight line divided harmonically by any conics P, Q envelopes a conic R, which touches the eight tangents to P, Q at the common points. For (assuming the envelope to be a conic) let α be a common tangent to R and P, then α touches P and is divided harmonically by P, Q; hence one of the points where it meets Q must coincide with its point of contact with P, or α touches P at one of the common points of P, Q. Now for two circles A, B, the envelope R will touch the tangents drawn to these circles at their common points; and hence, from the points at infinity, we see that the centres of the circles are the foci of the conic. If then A intersects B, C in real points at a finite distance, and a, b, c be the centres of the circles, draw a conic with foci a, b touching one (and therefore all) of the tangents to A, B at their common points; and a conic with foci a, c touching the tangents to A, C at their common points: the common tangents of these two will be the lines required. Since the conics have one common focus, they will have only two common tangents, besides the impossible two from the circular points. If the circles do not intersect, the straight lines divided harmonically must intersect both circles in impossible points; but it would be perfectly easy to find the equations.

2605. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—A horizontal beam AB, supported but not fixed at both ends A and B, is traversed from end to end by a moving load W distributed uniformly over a segment of it of constant length PQ. Show that the greatest bending moment M at any point X of the beam corresponds to the position of the load in its passage over the point for which PX : QX = AX : BX, and has for value

$$M = \frac{W}{AB^2} \cdot AX \cdot BX \cdot (AB - \tfrac{1}{2}PQ).$$

I. *Solution by* J. J. WALKER, M.A.

Let AX = a, BX = b, AB = $a + b = L$,
PQ = l, x = distance of any point between P and X from A, AP = x', y = distance of any point between Q and X from B, BQ = y'. Then the bending moment at X will be



$$M = \frac{W}{lL}\left(\int_{x'}^{a} bx\,dx + \int_{y'}^{b} ay\,dy\right) = \frac{W}{2lL}(abL - bx'^2 - ay'^2).$$

This will be a maximum when $bx\,dx' + ay\,dy' = 0$.
But $x' + y' = L - l$, whence $dx' + dy' = 0$. The condition for the maximum becomes, therefore, $bx = ay'$, or $a : x' = b : y'$, or $a : a - x' = b : b - y'$, or AX : PX = BX : QX, which proves the first part of the question.

Again, from the equations $bx' - ay' = 0$, $x' + y' = L - l$, we have

$$x = \frac{a}{L}(L-l), \quad y' = \frac{b}{L}(L-l),$$

and substituting these values in the general expression for M, the maximum moment becomes

$$\frac{W}{2lL}\,ab\left\{L - \frac{(L-l)^2}{L}\right\} = \frac{W}{L^2}\,ab\,(L - \tfrac{1}{2}l) \text{ or } \frac{W}{AB^2}\,AX \cdot BX\,(AB - \tfrac{1}{2}PQ).$$

II. *Solution by the* PROPOSER.

By known principles (see *Reprint*, Vol. IV., p. 29), if PQ lie altogether between A and X, then

$$M = \frac{W}{AB} \cdot \tfrac{1}{2}(AP + AQ) \cdot BX \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

if it lie altogether between B and X, then

$$M = \frac{W}{AB} \cdot \tfrac{1}{2}(BP + BQ) \cdot AX \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and if it lie partly between A and X, and partly between B and X, then

$$M = \frac{W}{AB} \cdot \left[\frac{PX}{PQ} \cdot (AX - \tfrac{1}{2}PX) \cdot BX + \frac{QX}{PQ} \cdot (BX - \tfrac{1}{2}QX) \cdot AX\right] \ldots\ldots (3),$$

which, since in that case PX + QX = PQ, becomes by reduction

$$M = \frac{W}{AB} \cdot \left[AX \cdot BX - \frac{1}{2 \cdot PQ} \cdot (BX \cdot PX^2 + AX \cdot QX^2)\right] \ldots\ldots (4).$$

The maxima values of M for these three cases, respectively, correspond, of course, for the first to the maximum value of $\frac{1}{2}(AP + AQ)$, that is to $AX - \frac{1}{2}PQ$, since then Q coincides with X; for the second to the maximum value of $\frac{1}{2}(BP + BQ)$, that is to $BX - \frac{1}{2}PQ$, since then P coincides with X; and for the third to the minimum value of $BX \cdot PX^2 + AX \cdot QX^2$, that is to $\frac{PQ^2}{AB} \cdot AX \cdot BX$, since then $BX \cdot PX = AX \cdot QX$, or, which is the same thing, PX : QX = AX : BX; thus giving, for the maxima values of M in the three cases respectively,

$$M = \frac{W}{AB} \cdot AX \cdot BX \cdot \left(1 - \frac{1}{2}\frac{PQ}{AX}\right), \text{ for the first case;}$$

$$M = \frac{W}{AB} \cdot AX \cdot BX \cdot \left(1 - \frac{1}{2}\frac{PQ}{BX}\right), \text{ for the second case;}$$

$$M = \frac{W}{AB} \cdot AX \cdot BX \cdot \left(1 - \frac{1}{2}\frac{PQ}{AB}\right), \text{ for the third case,}$$

which last, since AX + BX = AB, is, as above stated, the greatest of the three, and has evidently the value there given.

N.B.—For the two extreme cases in which PQ = 0 and = AB, the last formula gives, as of course it ought, the two well known values of M for those cases respectively, viz., $M = \frac{W}{AB} . AX . BX$, and $= \frac{1}{2} \frac{W}{AB} . AX . BX$. (See *Reprint*, Vol. IV., p. 30.)

2453. (Proposed by Professor WHITWORTH.)—The total number of partitions of n different things (*i.e.* the number of ways in which n different things can be distributed into 1, 2, 3, ... or n indifferent parcels) is the coefficient of x^n in the expansion of $\lfloor \underline{n} . \epsilon^{\epsilon^x - 1}$

Solution by the REV. J. WOLSTENHOLME, M.A.

By Quest. 2414 the number of ways of distributing n different things to r different people so that each may have at least one, is the coefficient of x^n in the expansion of $\lfloor \underline{n} (e^x + 1)^r$. Hence the number of ways of dividing the n things into r heaps is $(\lfloor \underline{r})^{-1}$ multiplied by the former number, since permuting the heaps gives no different partition. The whole number of partitions is therefore $\lfloor \underline{n}$ times the coefficient of x^n in the expansions of

$$(\epsilon^x - 1)^1 + \frac{1}{\lfloor \underline{2}} (\epsilon^x - 1)^2 + \ldots . + \frac{1}{\lfloor \underline{n}} (\epsilon^x - 1)^n,$$

and since the coefficients of x^n in subsequent terms are zero, we may consider this series to go on to infinity. The whole number is then the coefficient of x^n in the expansion of $\lfloor \underline{n} . \epsilon^{\epsilon^x - 1}$.

2639. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—The direction angles of three lines through the origin are $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$; required the equation of the cone of revolution of which they are sides.

Solution by J. J. WALKER, M.A.

Let (l, m, n), (l', m', n'), (l'', m'', n'') be the direction-cosines of the three given lines. The equation to a plane cutting them at points equidistant (by r) from the origin will be

$$\frac{L}{D} x + \frac{M}{D} y + \frac{N}{D} z = r,$$

where $D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ l & m & n & 1 \\ l' & m' & n & 1 \\ l'' & m'' & n'' & 1 \end{vmatrix}$, and L, M, N are the minors formed by omitting the second, third, and fourth rows and columns successively.

At the common section of this plane and the sphere

$$x^2 + y^2 + z^2 = r,$$

$$(L^2 - D^2)x^2 + (M^2 - D^2)y^2 + (N^2 - D^2)z^2 + 2MNyz + 2LNxz + 2LMxy = 0.$$

This is the equation to a cone having its vertex at the origin, passing through the three given lines, and having its sections by planes cutting these sides in points equidistant from the vertex circular. Hence it is the required cone.

2521. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—Two lines in space intersect a plane conic; show that the ruled surface determined by the three as directors is a quadric.

Solution by A. W. PANTON, B.A.

Describe a quadric through the two given lines, the two points on the conic consecutive to the points where the lines meet it, and any third point on the conic. This quadric is cut by the plane of the given conic in a conic having in common with the former three points, and the tangents at two of them. The conics are therefore coincident, and we have a quadric passing through the given lines and the given conic.

2630. (Proposed by W. S. BURNSIDE, M.A.)—Show that the degree of the locus of the foci of a system of conics, subject to any four conditions, is three times as great as that of the locus of their centres.

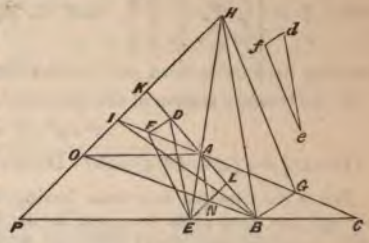
Solution by the PROPOSER.

Whenever a real focus goes off to infinity it is accompanied with the two imaginary foci, which take their place at the imaginary circular points, and the centre follows too, whence the line at infinity has three foci and one centre only *corresponding*, on it.

2607. (Proposed by W. HOPPS.)—In three straight lines given in position, to determine the positions of the vertices of a triangle of given or minimum magnitude, and of given species.

Solution by the PROPOSER.

Let A, B, C be the intersections of the three given lines in pairs, and suppose the vertices D, E, F of a triangle of the same magnitude and species as the given triangle *def*, to be posited in these lines as in the diagram. Draw lines from B respectively parallel to DE, DF, the former meeting the line through E, A at H, and the latter the line AC at G, and join G, H. Then, by parallels, we have



$$AH : AE = AB : AD = AG : AF;$$

hence the triangles BHG, DEF are similar. Suppose the circle passing through B, G, H to cut CA in I, and draw the lines BI, HI, the latter cutting BA in K. Now B, G, H, I being points in the circumference of a circle, the angles BIC, CIH are respectively equal to BHG, GBH, which are given angles; hence the point I and the position of the line HI are given. Let P be the intersection of HI, CB, and from A, E draw lines respectively parallel to DE, HP, to meet at N, the latter also meeting AB at L; and from B draw a line through N, meeting HP at O. Then we have

$$KB : LD = BH : DE = AH : AE = KA : LA,$$

$$\therefore\ KB : KA\,(1) = LD : LA = DE : AN\,(2) = LE : LN = KP : KO,$$

consequently the line joining O, A is parallel to BC; also from (1) and (2), AN is given in length. We have, therefore, the following

Construction.—Determine I the intersection with CA of the circular arc containing an angle equal to *def*, and whose chord is BC. Through I draw a line meeting CA at K, CB at P, and a line through A, parallel to BC, at O, and making with CI the angle KIC = *edf*. Draw OB, and from A inflect a line equal to a fourth proportional to KB, KA and *de*, to meet OB at N. Draw NE parallel to KP, and ED to AN; also make the angle DEF = *def* and join F, D. Then from the analysis, it is obvious that the triangle DEF, whose vertices are posited in the given lines, is of the same magnitude and species as the given triangle *def*.

Again, from (1) and (2) in the analysis, we have KB : KA = DE : AN. Whence it is evident that when DE, or the triangle DEF, is a minimum, AN is also a minimum; which is manifestly the case when AN is perpendicular to OB. Moreover, when the triangle DEF is *not* a minimum, it is clear that its vertices, collectively, may have two different positions in the given lines, inasmuch as the line AN may be inflected to meet OB at two different points. And finally, it may be remarked that, from the solution preceding, it may readily be shown that in three given lines the vertices of a triangle of given magnitude and species may have several different positions, and consequently each position has a corresponding triangle of minimum magnitude and of the same species; but, in the foregoing solution, it is not shown whether these minimum triangles are, or are not, of the same magnitude; however, when the given triangle is equiangular, it may be easily shown that there subsists but one minimum triangle of the same species.

[Another solution is given in Townsend's *Modern Geometry*, Vol. I., Art. 55.]

2482. (Proposed by H. R. GREER, B.A.)—A quadrilateral circumscribes an ellipse, the eccentric angles of the points of contact being α, β, γ, δ. Prove that the condition that a pair of opposite vertices should lie on a confocal conic may be thus expressed:—

$$a^2(\cos A+\cos B+\cos C+\cos D)(\cos A+\cos B-\cos C-\cos D)\times$$
$$(\cos C+\cos A-\cos B-\cos D)(\cos B+\cos C-\cos A-\cos D)=$$
$$b^2(\sin A+\sin B+\sin C+\sin D)(\sin A+\sin B-\sin C-\sin D)+$$
$$(\sin C+\sin A-\sin B-\sin D)(\sin B+\sin C-\sin A-\sin D);$$

where $A=\frac{1}{2}(-\alpha+\beta+\gamma+\delta)$, $B=\frac{1}{2}(\alpha-\beta+\gamma+\delta)$,
$C=\frac{1}{2}(\alpha+\beta-\gamma+\delta)$, $D=\frac{1}{2}(\alpha+\beta+\gamma-\delta)$.

Solution by the PROPOSER.

Let tangents be drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points whose eccentric angles are α and β; then the coordinates of the intersection of these are $x=a\frac{\cos\epsilon}{\cos\eta}$, $y=b\frac{\sin\epsilon}{\cos\eta}$, where $\epsilon=\frac{1}{2}(\alpha+\beta)$ and $\eta=\frac{1}{2}(\alpha-\beta)$. The condition that this point should lie upon the confocal ellipse $\frac{x^2}{a^2+h^2}+\frac{y^2}{b^2+h^2}=1$ is

$$a^2(b^2+h^2)\cos^2\epsilon+b^2(a^2+h^2)\sin^2\epsilon=(a^2+h^2)(b^2+h^2)\cos^2\eta.$$

Similarly, the condition that the intersection of tangents at the points defined by the eccentric angles γ and δ should lie on the same confocal is, putting $\upsilon=\frac{1}{2}(\gamma+\delta)$ and $\omega=\frac{1}{2}(\gamma-\delta)$,

$$a^2(b^2+h^2)\cos^2\upsilon+b^2(a^2+h^2)\sin^2\upsilon=(a^2+h^2)(b^2+h^2)\cos^2\omega.$$

Let us eliminate h^2 from these two equations. Multiplying crossways, we get an equation which is equivalent to the following pair:

$$\left.\begin{aligned}a^2(b^2+h^2)&=AS\\b^2(a^2+h^2)&=AK\end{aligned}\right\},\quad\text{where}\quad\left\{\begin{aligned}S&=\sin^2\upsilon\cos^2\eta-\sin^2\epsilon\cos^2\omega\\K&=\cos^2\epsilon\cos^2\omega-\cos^2\eta\cos^2\upsilon\end{aligned}\right\};$$

after which we easily get the eliminant in the form

$$(S\cos^2\epsilon+K\sin^2\epsilon)(a^2K-b^2S)=(a^2-b^2)SK\cos^2\eta;$$

or, putting $T=\sin^2\upsilon\cos^2\epsilon-\cos^2\upsilon\sin^2\epsilon$, in the form

$$a^2K(T-S)=b^2S(T-K),\quad i.e.,\text{ in the form}$$

$$a^2(\cos\epsilon\cos\omega+\cos\eta\cos\upsilon)(\cos\epsilon\cos\omega-\cos\eta\cos\upsilon)$$
$$\times(\sin\upsilon\sin\eta+\sin\epsilon\sin\omega)(\sin\upsilon\sin\eta-\sin\epsilon\sin\omega)$$
$$=b^2(\sin\upsilon\cos\eta+\sin\epsilon\cos\omega)(\sin\upsilon\cos\eta-\sin\epsilon\cos\omega)$$
$$\times(\sin\omega\cos\epsilon+\sin\eta\cos\upsilon)(\sin\omega\cos\epsilon-\sin\eta\cos\upsilon),$$

observing that $2\cos\epsilon\cos\omega=\cos(\epsilon+\omega)+\cos(\epsilon-\omega)$, and applying analogous transformation to the other terms, we deduce the required result. In it α, β, γ, δ are interchangeable; which proves the theorem (Quest. 1819) that if a quadrilateral circumscribe an ellipse having one pair of its opposite vertices on a confocal conic, each other pair will also lie on a confocal conic.

2465. (Proposed by the late R. H. WRIGHT, M.A.)—If the tangent drawn to the ellipse $a^2y^2+b^2x^2=a^2b^2$, at one end of a chord whose pole is (h, k), be represented by $a^2yy'+b^2xx'=a^2b^2$, find the equation to the tangent at the other end of the polar.

Solution by W. H. LAVERTY, B.A.

Let (x_1, y_1) be the other extremity of the polar, and let $b^2h^2+a^2k^2-a^2b^2=z^4$; then, for the intersections of the polar of (h, k) with the curve, we have

$$x_1=\frac{a^2}{b^2h^2+a^2k^2}(b^2h+z^2k);\quad x'=\frac{a^2}{b^2h^2+a^2k^2}(b^2h-z^2k);$$

$$y_1=\frac{b^2}{b^2h^2+a^2k^2}(a^2k-z^2h);\quad y'=\frac{b^2}{b^2h^2+a^2k^2}(a^2k+z^2h);$$

therefore $x_1=x'\cdot\dfrac{b^2h+z^2k}{b^2h-z^2k};\qquad y_1=y'\cdot\dfrac{a^2k-z^2h}{a^2k+z^2h}.$

Hence the equation to the tangent at (x_1, y_1) is $a^2yy_1+b^2xx_1=ab$, which, by substituting and multiplying out, becomes

$$(a^2k-z^2h)(b^2h-z^2k)a^2yy'+(a^2k+z^2h)(b^2h+z^2k)b^2xx'$$
$$=(a^2k+z^2h)(b^2h-z^2k)a^2b^2.$$

2425. (Proposed by T. T. WILKINSON, F.R.A.S.)—A heavy cone stands inverted on its vertex, its axis making an angle α with the horizon; find the magnitude and direction of the least power which, when applied at the centre of its base, will just sustain the cone in that position.

Solution by J. J. WALKER, M.A.

Let P be the required force, Q the resultant of the pressures on the vertex of the cone, W its weight. These three forces will lie in the vertical plane passing through the axis of the cone. Resolving them horizontally and vertically, and taking the moments with respect to the vertex, the three conditions of equilibrium give, θ and ϕ being the angles which the directions of P and Q respectively make with the horizontal,

$$P\cos\theta+Q\cos\phi=0\ldots\ldots\ldots(1),$$
$$W+P\sin\theta+Q\sin\phi=0\ldots\ldots\ldots(2),$$
$$\tfrac{3}{4}W\cos\alpha+P\sin(\alpha+\theta)=0\ldots\ldots\ldots(3).$$

From (3), $P=-\dfrac{3}{4}\dfrac{\cos\alpha}{\sin(\alpha+\theta)}W$, and will therefore be least when $\theta=90°-\alpha$, or P lies in the base of the cone, its value being then

$$P=-\tfrac{3}{4}\cos\alpha\,.\,W\ldots\ldots\ldots(4),$$

From (1) and (2), $\tan\phi = \dfrac{W + P\sin\theta}{P\cos\theta} = -\dfrac{1}{8}\dfrac{1+3\sin^2\alpha}{\sin\alpha\cos\alpha}$ or $-\dfrac{1}{8}(4\tan\alpha + \cot\alpha)$,

when P is the least force; whence $\cos\phi = \dfrac{-3\sin\alpha\cos\alpha}{\sqrt{(1+15\sin^2\alpha)}}$.

Finally, from (1), $Q = \dfrac{-P\cos\theta}{\cos\phi} = -\dfrac{W}{4}\sqrt{(1+15\sin^2\alpha)}$.

2382. (Proposed by H. R. GREER, B.A.)—Given a quartic curve Q, and a point on it P; suppose that from p there are drawn the ten tangents to Q, omitting that at P, touching Q in ten points (t) and meeting it again in ten points (r). The t points, we know, lie on a cubic K; prove that the r points lie on a quartic U, with a point of osculation at p, touching K at p (counting for four points of intersection), and in four other points.

Solution by JAMES DALE.

The equation which gives the points where the line meets the curve which joins the given point p (x_1, y_1, z_1) to a variable point (x, y, z), is

$$l^4Q + l^3m(\Delta_1Q) + \tfrac{1}{2}l^2m^2(\Delta_1^2Q) + lm^3(\Delta Q_1) + m^4Q_1 = 0,$$

where Q, Δ_1Q, Δ_1^2Q, ΔQ_1, Q_1 are respectively of the 4th. 3rd, 2nd, 1st, and 0th degree in x, y, z. If the point (x_1, y_1, z_1) lie on $Q=0$, then $Q_1=0$, and the equation becomes $l^3Q + l^2m\Delta_1 + \tfrac{1}{2}lm^2\Delta_1^2 + m^3\Delta = 0$

(writing Δ_1, Δ_1^2, Δ instead of Δ_1Q, Δ_1^2Q, ΔQ_1). The condition that this equation should have two equal roots is

$$\left\{27\Delta^2Q - 9\Delta\Delta_1\Delta_1^2 + \tfrac{1}{2}(\Delta_1^2)^3\right\}Q = \left\{\tfrac{1}{4}(\Delta_1^2)^2 - 4\Delta\Delta_1\right\}(\Delta_1)^2.$$

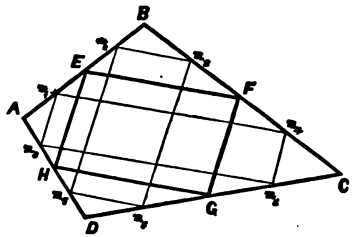
The form of the equation shows that it represents a locus of the 10th degree, touching Q in the points where Q meets Δ_1. The other points where Q meets the locus lie on the curve $(\Delta_1^2)^2 - 16\Delta\Delta_1 = 0$. Hence the points of contact of the ten tangents drawn from p on the quartic Q lie on the cubic $\Delta_1 = K = x_1\dfrac{dQ}{dx} + y_1\dfrac{dQ}{dy} + z_1\dfrac{dQ}{dz} = 0$, and the ten points where the ten tangents meet Q again lie on the quartic $(\Delta_1^2)^2 - 16\Delta\Delta_1 = U = 0$; and the form of the equation of U shows that it has double contact with K in the points where K meets the conic $\Delta_1^2 = 0$, and as K and Δ_1^2 have two coincident points at p (x_1, y_1, z_1), it follows that K and U have coincident points at p.

2499. (Proposed by S. WATSON.)—ABCD is a quadrilateral; E, F, G, H the middle points of AB, BC, CD, DA; and $m_1, m_2, \ldots m_8$ the middle points of AE, EB, BF, ... HA. Show that the sum of the squares of the 28 lines joining every two of the points $m_1, m_2, \ldots m_8$ is

$$4(AC^2 + BD^2) + AB^2 + BC^2 + CD^2 + DA^2.$$

Solution by R. TUCKER, M.A.; S. BILLS; *and others.*

Denoting all such lines as m_1m_2 by $\lambda_1, \lambda_2 \ldots \lambda_8$; as m_1m_3 by $\mu_1, \mu_2, \ldots \mu_8$; as m_1m_6 by $\nu_1, \ldots \nu_4$; we obtain, by known properties of triangles, bearing in mind that the figures EG_1, m_1m_5, m_2m_6 are parallelograms,



$$\nu_1^2 + m_1m_5^2 = 2(m_1G^2 + m_6G^2),$$

and $\nu_2^2 + m_2m_6^2 = 2(m_2G^2 + m_6G^2)$;

therefore $\nu_1^2 + \nu_2^2 + m_1m_5^2 + m_2m_6^2 = \lambda_1^2 + \lambda_5^2 + 4EG^2$ }

and $\nu_3^2 + \nu_4^2 + m_3m_7^2 + m_4m_8^2 = \lambda_3^2 + \lambda_7^2 + 4FH^2$ } ..... (α).

Again, $\mu_1^2 = \lambda_1^2 + 2Bm_2^2 + 3\lambda_2^2 - 2Bm_3^2$ }

and $\mu_2^2 = \lambda_3^2 + 2Bm_3^2 + 3\lambda_2^2 - 2Bm_2^2$ } ;

therefore $\mu_1^2 + \mu_2^2 = \lambda_1^2 + \lambda_3^2 + 6\lambda_2^2$;

hence $\Sigma(\mu^2) = 2(\lambda_1^2 + \lambda_3^2 + \lambda_5^2 + \lambda_7^2) + 6(\lambda_2^2 + \lambda_4^2 + \lambda_6^2 + \lambda_8^2)$ ..... (β).

Now $(\alpha) + (\beta) = 3(\lambda_1^2 + \lambda_3^2 + \lambda_5^2 + \lambda_7^2) + 44(\lambda_2^2 + \lambda_8^2)$;

and the twelve remaining squares (γ) are easily shown to be equal to

$$\lambda_1^2 + \lambda_3^2 + \lambda_5^2 + \lambda_7^2 + 20(\lambda_2^2 + \lambda_8^2).$$

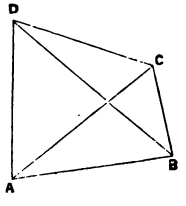
Hence the sum of the whole number of squares required is

$$(\alpha) + (\beta) + (\gamma) = 4(\lambda_1^2 + \lambda_3^2 + \lambda_5^2 + \lambda_7^2) + 64(\lambda_2^2 + \lambda_8^2)$$
$$= AB^2 + BC^2 + CD^2 + DA^2 + 4(AC^2 + BD^2).$$

1971. (Proposed by C. W. MERRIFIELD, F.R.S.)—In a given tetrahedron, to find a set of rectangular coordinate axes, such that each axis shall pass through a pair of opposite edges.

Solution by J. J. WALKER, M.A.

Take the three edges AB, AC, AD of the tetrahedron as axes of x, y, z respectively, and let $AB = x'$, $AC = y'$, $AD = z'$. If ξ, η, ζ be the coordinates of any point (O), the plane determined by this point and the line CD will have as its equation



$$x(y'z' - \eta z' - \zeta y') + y\xi z' + z\xi y' - \xi y'z' = 0 \ldots (1),$$

since this equation is evidently satisfied by the coordinates of O, C, and D. Again, the plane determined by O and AB will have as its equation

$$y\zeta - z\eta = 0 \ldots (2).$$

Hence, if the line passing through O and intersecting the opposite edges AB, CD be $\frac{x-\xi}{l} = \frac{y-\eta}{m} = \frac{z-\zeta}{n}$, we shall have

$$l : m : n = \xi(\eta z' + \zeta y') : \eta(\eta z' + \zeta y' - y'z') : \zeta(\eta z' + \zeta y' - y'z').$$

And, by symmetry, if $\frac{x-\xi}{l'} = \frac{y-\eta}{m'} = \frac{z-\zeta}{n'}$ be the line passing through O and intersecting the opposite edges AC and BD,

$$l' : m' : n' = \xi(\xi z' + \zeta x' - x'z') : \eta(\xi z' + \zeta x') : \zeta(\xi z' + \zeta x' - x'z').$$

If now λ, μ, ν be the cosines of the angles CAD, BAD, BAC respectively, the conditions that the above two lines passing through O should be perpendicular one to the other is

$$ll' + mm' + nn' + \lambda(mn' + m'n) + \mu(l'n + ln') + \nu(lm' + l'm) = 0;$$

or, substituting from the above proportions, the condition is

$$\begin{aligned} 0 = {} & \xi^2(\xi z' + \zeta x' - x'z')(\eta z' + \zeta y') + \eta^2(\eta z' + \zeta y' - y'z')(\xi z' + \zeta x') \\ & + \zeta^2(\xi z' + \zeta x' - x'z')(\eta z' + \zeta y' - y'z') + \lambda\eta\zeta(2\xi z' + 2\zeta x' - x'z')(\eta z' + \zeta y' - y'z') \\ & + \mu\xi\zeta(2\eta z' + 2\zeta y' - y'z')(\xi z' + \zeta x' - x'z') \\ & + \nu\xi\eta\left\{(\xi z' + \zeta x')(\eta z' + \zeta y') + (\xi z' + \zeta x' - x'z')(\eta z' + \zeta y' - y'z')\right\} \quad \ldots (3). \end{aligned}$$

If it be desired to express this condition in terms of the six edges only, we may substitute, for λ, $\frac{y'^2 + z'^2 - x''^2}{2y'z'}$, for μ, $\frac{x'^2 + z'^2 - y''^2}{2x'z'}$, for ν, $\frac{x'^2 + y'^2 - z''^2}{2x'y'}$, supposing $CD = x''$, $BD = y''$, $BC = z''$.

The condition (3) may be thrown into any one of the forms

$$\begin{aligned} 0 = {} & (\xi^2 + \eta^2 + \zeta^2 + 2\lambda\eta\zeta + 2\mu\xi\zeta + 2\nu\xi\eta)(\xi z' + \zeta x')(\eta z' + \zeta y') \\ & - (\xi^2 + \zeta^2 + \lambda\eta\zeta + 2\mu\xi\zeta + \nu\xi\eta)(\eta z' + \zeta y')x'z' \\ & - (\eta^2 + \zeta^2 + 2\lambda\eta\zeta + \mu\xi\zeta + \nu\xi\eta)(\xi z' + \zeta x')y'z' \\ & + (\zeta^2 + \lambda\eta\zeta + \mu\xi\zeta + \nu\xi\eta)x'y'z'^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4), \end{aligned}$$

$$\begin{aligned} 0 = {} & \xi(\xi + \nu\eta + \mu\zeta)(\xi z' + \zeta x' - x'z')(\eta z' + \zeta y') \\ & + \eta(\nu\xi + \eta + \lambda\zeta)(\eta z' + \zeta y' - y'z')(\xi z' + \zeta x') \\ & + \zeta(\mu\xi + \lambda\eta + \zeta)(\xi z' + \zeta x' - x'z')(\eta z' + \zeta y' - y'z') \\ & + \nu\left\{(\xi z' + \zeta x')y' + (\eta z' + \zeta y')x' + x'y'z'\right\}\xi\eta z' \quad \ldots\ldots (5), \end{aligned}$$

$$\begin{aligned} 0 = {} & x'y'\zeta^4 + \left\{(\xi y' + \eta x')z' + 2(\mu\xi + \lambda\eta - z')x'y'\right\}\zeta^3 \\ & + \left\{(\xi - x')(\eta - y')z'^2 + (\xi^2 + \eta^2)x'y' + (\mu\xi + \lambda\eta)(2\xi y' + 2\eta x' - 3x'y')z'\right\}\zeta^2 \\ & + \left\{(\xi^2 + \eta^2 + 2\nu\xi\eta)(\xi y' + \eta x' - x'y')z' + [(\mu\xi + \lambda\eta)(2\xi\eta + x'y' - \xi y' - \eta x') \right. \\ & \qquad \left. - (\mu x' + \lambda y')\xi\eta]z'^2\right\}\zeta \\ & + \left\{\xi(\xi - x') + \eta(\eta - y') + \nu[(\xi - x')(\eta - y') + \xi\eta]\right\}\xi\eta z'^2 \quad \ldots\ldots\ldots (6). \end{aligned}$$

It appears from the above investigation that the locus of a point determined by the condition that two lines meeting in it shall be perpendicular one to the other (or, generally, be inclined at any given angle), these lines resting, one on two fixed lines in space, the other on two other fixed lines intersecting each one of the former pair, is a quartic surface. Hence the origin of the three rectangular axes referred to in the question may be any one of the points common to three quartic surfaces; viz., that to which (4) is the equation, and two others whose equations are found from (4) by interchanging ξ and ζ, η and ζ, successively.

Four points common to these surfaces are, plainly, the four corners of the tetrahedron. The directions of the three axes drawn through the point A will be found by determining ll'', mm', $n'n''$ from the six equations

$$\left.\begin{aligned} mm' + \lambda mn' + \mu ln' + \nu lm' &= 0 \\ ll'' + \lambda mn'' + \mu ln'' + \nu l''m &= 0 \\ n'n'' + \lambda m'n'' + \mu l''n' + \nu l''m' &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} l^2 + m^2 + 2\nu lm - 1 &= 0 \\ m'^2 + n'^2 + 2\lambda m'n' - 1 &= 0 \\ l''^2 + n''^2 + 2\mu l''n'' - 1 &= 0 \end{aligned}\right\},$$

the equations to the axes being $\frac{x}{l} = \frac{y}{m}$, $\frac{y}{m'} = \frac{z}{n'}$, $\frac{z}{n''} = \frac{x}{l''}$.

In this case, the construction of the three axes may be conceived geometrically thus:—At A raise a perpendicular P to the plane ABC, meeting a sphere with centre A and radius AD in the point Q; through P two planes may be drawn, each cutting the planes ADC, ABD in a pair of rectangular right lines, (for the solution of this problem would be effected by drawing through Q a great circle of the sphere such that its arc intercepted between the circles in which the planes ADC, ABD cut the sphere may be a quadrant). A right line in the plane ABC, perpendicular to either of these planes through P, would form the third rectangular axis to the corresponding pair in the planes ACD, ABD.

[A Solution by *Quaternions* is given on p. 85 of Vol. VIII. of the *Reprint*.]

2696. (Proposed by the Rev. T. P. KIRKMAN, M.A., F.R.S.)—Let $G = 1 + \theta + \theta^2 + \theta^3$ be any group of powers of a substitution made with the elements 1234. Let H_1 be any asymmetric function of $x_1x_2x_3x_4$. And let $\theta\,;H_1$ or $G\,;H_1$ be the symbol of the result of operation on H_1 by θ or by the whole group G. Thus $G\,;H_1$ is an algebraic sum of four values of H_1.

1. There is a substitution ϕ such that $\left\{(G\,;H_1^i)^j + \phi\,;(G\,;H_1^i)^j\right\}^k$ is a function which has not more than three values, all given with G and H_1, whatever be the integer exponents i, j, k.

2. Hereby the algebraic solution of the general quintic equation is attainable at the cost of few words.

Solution by the PROPOSER.

Let the roots of the quintic $\Theta = 0$ be $x_0x_1x_2x_3x_4$. We can write, if α be an imaginary fifth root of unity,

$$x_0 = \Sigma x + (\alpha + \alpha^2 + \alpha^3 + \alpha^4)(x_1 + x_2 + x_3 + x_4), \text{ or}$$

$$x_0 = \tfrac{1}{5}\Sigma x + \tfrac{1}{5}(x_0 + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3 + \alpha^4 x_4)^{\frac{5}{5}} + \tfrac{1}{5}(x_0 + \alpha x_2 + \alpha^2 x_3 + \alpha^3 x_4 + \alpha^4 x_1)^{\frac{5}{5}},$$
$$+ \tfrac{1}{5}(x_0 + \alpha x_3 + \alpha^2 x_4 + \alpha^3 x_1 + \alpha^4 x_2)^{\frac{5}{5}} + \tfrac{1}{5}(x_0 + \alpha x_4 + \alpha^2 x_1 + \alpha^3 x_2 + \alpha^4 x_3)^{\frac{5}{5}},$$

which, by ordinary involution, becomes

$$x_0 = \tfrac{1}{5}\Sigma x + \tfrac{1}{5}(S + H_1)^{\frac{1}{5}} + \tfrac{1}{5}(S + H_2)^{\frac{1}{5}} + \tfrac{1}{5}(S + H_3)^{\frac{1}{5}} + \tfrac{1}{5}(S + H_4)^{\frac{1}{5}},$$

where S is symmetric in $x_0x_1x_2x_3x_4$, and H_1 is a 24-valued function of the form

$$H_1 = u_0 + \alpha u_1 + \alpha^2 u_2 + \alpha^3 u_3 + \alpha^4 u_4,$$

which is invariable by the cyclical permutation of $x_0x_1x_2x_3x_4$, and consequently receives all its 24 values by the permutations of any four of them, for example by those of $x_1x_2x_3x_4$. H_2, H_3, and H_4 are made from H_1 by the cyclical permutations of $x_1x_2x_3x_4$, so that, if G be the group 1234, 2341, 3412, 4123, we have $G;H_1 = H_1 + H_2 + H_3 + H_4$,

the root x_0 being undisturbed by the substitutions of G_1; that is, $G;H$ is a function of x_1, x_2, x_3, x_4, invariable by the substitutions of G, and equally so is

$$G;H_1^i = H_1^i + H_2^i + H_3^i + H_4^i = K_i,$$

whatever be the integer i; for the substitution which changes H_n into H_m will change $H_n . H_n$ into $H_m . H_m$ and H_n^i into H_m^i. And it is evident that

$$(G;H_1^i)^j = (H_1^i + H_2^i + H_3^i + H_4^i)^j = K_i^j$$

is invariable by the substitutions of G; for all its j factors are so: in other words, K_i^j is a function formed upon the group G, of the fourth order. The group of the eighth order $P = G + \phi;G = \begin{matrix}1234+1432 \\ 2341 \quad 4321 \\ 3412 \quad 3214 \\ 4123 \quad 2143\end{matrix} = G + {}_1G$ is composed of G and its derivate ${}_1G$ by any one ϕ of the four substitutions in ${}_1G$;

and
$$K_i^j + \phi;K_i^j = K_i^j + {}_1K_i^j = L_j$$
$$= (H_1^i + H_2^i + H_3^i + H_4^i)^j + (H_5^i + H_6^i + H_7^i + H_8^i)^j,$$

in which eight of the 24 values of H_1 appear, is a function formed on the group P and invariable by its eight substitutions.

This follows from a property of every group $R = G + {}_1G + {}_2G + \ldots + {}_mG$ composed of a group G of any order and of m derivates of G, that if θ be any substitution of the group R, the operation $\theta;R$ can change nothing in the right member except the order of $G, {}_1G \ldots {}_mG$; whence it follows that, if $(G), ({}_1G), \ldots ({}_mG)$ be functions formed on G and its m derivates, the function

$$(R)_i = (G)^i + ({}_1G)^i + \ldots + ({}_mG)^i$$

is invariable by the substitutions of R, and any power of (R_i) is the like; that is,
$$(K_i^j + {}_1K_i^j)^k = L_j^k$$

is a function invariable by the 8 substitutions of P, whatever be the integers k, j, i.

It follows that L_j^k is a three-valued function; for if $\lambda = 2314$ and $\lambda^2 = 3124$,

$$\Omega = P + \lambda;P + \lambda^2;P = P + {}_1P + {}_2P$$

is the whole group of 1.2.3.4 substitutions made with 1234. The sum of the three values of L_j^k made on P and its two derivates ${}_1P$ and ${}_2P$, viz.

$$\Sigma_k = L_j^k + {}_1L_j^k + {}_2L_j^k = L_j^k + \lambda;L_j^k + \lambda^2;L_j^k,$$

is therefore invariable by the 24 substitutions of Ω; that is, Σ_k is a function of $x_0x_1x_2x_3x_4$, which is symmetrical in the four roots $x_1x_2x_3x_4$: it is also a function of the 24 values of H_1, every one of which is invariable by the

cyclical permutations of $x_0x_1x_2x_3x_4$; but this does not in general make Σ_k invariable by them. Σ_k is in fact a five-valued function of $x_0x_1x_2x_3x_4$.

The function Σ_k may be thus symbolized,

$$\Sigma_k = G_3\,;\, G_2^k\,;\, G_4^j\,;\, H_1^i,$$

where G_4; $H_1^i = K_i^j$ is the jth power of the sum of four values of H_1^i formed on G above written of the 4th order; where $G_2^k\,;G_4^j\,;H_1^i = G_2^k\,;K_i^j = L_j^k$ is the kth power of the sum of two values of K_i^j made on $G_2 = 1234+1432$ of the second order, and where Σ_k is the sum of the three values of L_j^k formed on the group $G_3 = \begin{matrix}1234\\2314\\3124\end{matrix}$ of the third order. Thus Σ_k is the sum of the kth powers of three values of L_j, the sum of the jth powers of two values of K_i, the sum of the ith powers of four values of our given function H_1.

I find that it is more convenient for higher applications to form the function

$$\Sigma_\beta = \Gamma_4\,;\Gamma_3^\beta\,;\Gamma_2^\alpha\,;\,H_1^i = \Gamma_4\,;\,Z_\alpha^\beta,$$

where Γ_4 is the above written G, Γ_3 is the above G_3, and Γ_2 is $1234+2143$; *i. e.*, Γ_r is the group of cyclical permutations of $1, 2, \ldots r$. This Σ_β is the sum of the βth powers of four values of $Z_\alpha = \Gamma_3\,;\,Y_i^\alpha$, which is the sum of αth powers of three values of $Y_i = \Gamma_2\,;\,H_1^i$, which is the sum of ith powers of two values of our H_1. It follows directly from the above stated property of any group R, that Σ_β is a symmetrical function of $x_1x_2x_3x_4$, and also that

$$\Sigma_\mu = \Gamma_n\,;\,\Gamma_{n-1}^\mu\,;\,\Gamma_{n-2}^\lambda \cdots \Gamma_4^\gamma\,;\,\Gamma_3^\beta\,;\,\Gamma_2^\alpha\,;\,H_1^i$$

is a symmetrical function of $x_1x_2x_3 \ldots x_n$, if H_1 be any asymmetric function of these variables, whatever be the positive exponents $\mu, \lambda, \ldots \gamma, \beta, \alpha, i$.

Let Σ_β be formed for the 24 combinations of $i<3$, $\alpha<4$, $\beta<5$. We can write, for any fixed pair of values of α and i (there are six pairs),

$$\begin{matrix}\Sigma_1 = Z_\alpha + {}_1Z_\alpha + {}_2Z_\alpha + {}_3Z_\alpha & \Sigma_3 = Z_\alpha^3 + {}_1Z_\alpha^3 + {}_2Z_\alpha^3 + {}_3Z_\alpha^3\\ \Sigma_2 = Z_\alpha^2 + {}_1Z_\alpha^2 + {}_2Z_\alpha^2 + {}_3Z_\alpha^2 & \Sigma_4 = Z_\alpha^4 + {}_1Z_\alpha^4 + {}_2Z_\alpha^4 + {}_3Z_\alpha^4\end{matrix}$$

of which the first line shows four values of Z_α, the second four values of Z_α^2, &c., made on Γ_4.

The first members are all rational functions of $x_0x_1x_2x_3x_4$, and symmetrical in $x_1x_2x_3x_4$, the separate terms on the right are asymmetric functions of $x_0x_1x_2x_3x_4$. These four conditions enable us to transform Z_α into the *issuing root* $\left\{Z_\alpha\right\}$ of a quartic whose coefficients are given rational and symmetrical functions of $x_1x_2x_3x_4$, and the asymmetric Z_α is thus changed into an irrational function of $x_0x_1x_2x_3x_4$, which is symmetric in $x_1x_2x_3x_4$.

This $\left\{Z_\alpha\right\}$ we can thus obtain both for $i=1$ and for $i=2$; and, for a fixed value of i, we can write

$$\left\{Z_1\right\} = Y_i + {}_1Y_i + {}_2Y_i, \quad \left\{Z_2\right\} = Y_i^2 + {}_1Y_i^2 + {}_2Y_i^2, \quad \left\{Z_3\right\} = Y_i^3 + {}_1Y_i^3 + {}_2Y_i^3,$$

where we read on the right the values of Y_i^a formed on Γ_3. Hence Y_i can be written as the issuing root $\{Y_i\}$ of a given cubic, whose coefficients are given irrational functions of $x_0x_1x_2x_3x_4$, symmetrical in $x_1x_2x_3x_4$; and if we take care that the issuing roots $\{Z_1\}$, $\{Z_2\}$, $\{Z_3\}$ are corresponding roots of the quartics which yield them, the coefficients of this cubic will be unambiguous. We have two such cubics to solve, whose corresponding roots are

$$\{Y_1\} = H_1 + {}_1H_1, \quad \{Y_2\} = H_1^2 + {}_1H_1^2,$$

and H_1 is finally transformed into $\{H_1\}$, the root of a given quadratic, whose coefficients are irrational and unambiguous functions of $x_0x_1x_2x_3x_4$, symmetrical in $x_1x_2x_3x_4$. Let us write this $H_1 = \{{}_0H_1\}$, showing that H_1 is transformed into an irrational symmetric in all the roots except x_0.

The function H_1 of 24 values receives them all by the permutations of any four roots. In the preceding process, we have left x_0 undisturbed, treating it as a constant in our substitutions. We can repeat the process, and permuting only $x_0x_2x_3x_4$, leave x_1 undisturbed, forming our groups with 2340 instead of 1234. The result will be $H_1 = \{{}_1H_1\}$, or the same asymmetric H_1 is transformed into an irrational function of $x_0x_1x_2x_3x_4$, symmetrical in all except x_1. And we can thus obtain three more results

$$H_1 = \{{}_2H_1\}, \quad H_1 = \{{}_3H_1\}, \quad H_1 = \{{}_4H_1\},$$

where $\{{}_rH_1\}$ is symmetrical in all the roots except x_r. From these we deduce $\quad H_1^5 = \{{}_0H_1\}\cdot\{{}_1H_1\}\cdot\{{}_2H_1\}\cdot\{{}_3H_1\}\cdot\{{}_4H_1\} = [H_1],$

which is symmetrical in all the five roots $x_0x_1x_2x_3x_4$, and $H_1 = [H_1]^{\frac{1}{5}}$ is the required transformation of H_1 into a symmetrical function of the roots of $\Theta=0$. With equal ease we can obtain, by our five-fold process, the transformations $\quad H_2 = [H_2]^{\frac{1}{5}}, \quad H_3 = [H_3]^{\frac{1}{5}}, \quad H_4 = [H_4]^{\frac{1}{5}};$

and it is highly probable that we shall obtain four values at once in each process, merely by varying the reading of the radicals in that root $\{Z_a\}$ of the quartic that we chose out of four.

Wherefore nothing but a most monstrous amount of algebraic labour can prevent our writing the root of $\Theta=0$ thus, in terms of $x_0x_1x_2x_3x_4$,

$$x_0 = \tfrac{1}{5}\Sigma x + \tfrac{1}{5}(S + [H_1]^{\frac{1}{5}})^{\frac{1}{5}} + \tfrac{1}{5}(S + [H_2]^{\frac{1}{5}})^{\frac{1}{5}} + \tfrac{1}{5}(S + [H_3]^{\frac{1}{5}})^{\frac{1}{5}} + \tfrac{1}{5}(S + [H_4]^{\frac{1}{5}})^{\frac{1}{5}},$$

which is an irrational and a symmetrical function of all the roots.

Is this now an algebraical solution of the quintic $\Theta=0$? It is, or it is not, according as the irrationals $[H_1]$. $[H_2]$, $[H_3]$, $[H_4]$ are, or are not, for the reason that they are symmetrical in the five roots, reducible in a finite form into functions of the coefficients of $\Theta=0$.

If this reduction can be effected, as I believe it can, whatever be the number of the roots $x_0, x_1, \ldots x_n$, then assuredly the great problem of equations is solved for all degrees.

The five groups of 24 which we employ in the five processes for obtaining the quadratic roots, $\{{}_0H_1\}$, $\{{}_1H_1\}$, $\{{}_2H_1\}$, $\{{}_3H_1\}$, $\{{}_4H_1\}$, transforma-

tions of the same H_1, are got by cyclically permuting $x_0x_1x_2x_3x_4$ in any of the five groups; and they complete together the entire group of 120, each having one undisturbed element. Hence there is little doubt that from $\{_0H_1\}$ the remaining four irrationals will be got by cyclically permuting the five roots, without repetition of our process; while the five irrationals $\{_0H_2\}$, $\{_1H_2\}$, $\{_2H_2\}$, $\{_3H_2\}$, $\{_4H_2\}$, as well as two more fives, will be obtained by varying the reading of our quartic root $\{Z_a\}$. The five tactical values of $\{_0H_1\}$, being all numerically the same, lead to the solution of the binomial quintic $H_1^5 = [H_1]$, and the prognostic of the acute MURPHY and others is fulfilled for equations of all degrees.

The preceding investigation, even if a doubt remains, will, I hope, justify my proposal of Question 2696.

2511. (Proposed by Professor SYLVESTER.)—If on an ordinary chessboard we compare any two rows or any two columns, squares opposite one another will be all of unlike or all of like colours according as the number of rows or columns intervening is even (zero included) or odd.

1. Show that it is possible to paint the squares in a manner such that, on comparing any two rows or columns, four of the squares opposite one another shall have like and the remaining four unlike colours.

2. Extend the construction to the case of a chessboard containing 2^i (instead of eight) squares in each direction.

Solution by G. A. OGILVIE.

The principles upon which I have made this depend are as follows:—

(1.) There are only two ways of painting a chessboard containing four squares, so that, on comparing either of the two pairs of columns, there shall be one like and one unlike colour; viz., either with three white and one black or with three black and one white.

(2.) This property can be extended to the case of a board containing any number of squares when the board is made up of three similar groups of squares and one dissimilar group.

[By "group" is meant the fourth part of a square, made by joining middle points of sides; and by "dissimilar," that all the colours are changed so that the blacks become whites and the whites blacks.]

(3) If in a square, as mentioned in (1), containing two in each side, we consider that the first vertical column a and the second vertical column b should become $-a$ and $-b$ when the colours in each column are changed; then we can form a square of four in each side of the required species, containing a and $-b$ in the 1st column, b and $-a$ in the 2nd, $-a$ and $-b$ in the 3rd, b and a in the 4th; and this is evidently symmetrical with regard

to its horizontal columns. Let B stand for a black and W for a white square;

then $\begin{matrix}B\\B\end{matrix}$ is the column represented by a, or $\begin{matrix}W\\W\end{matrix}$, or &c.

and $\begin{matrix}W\\W\end{matrix}$ is the column represented by $-a$, or $\begin{matrix}B\\B\end{matrix}$, or &c.

The two squares of four in each side, obtained by the processes indicated in (2) and (3) are

(Fig. 1)
B W B W
B B B B
B W W B
B B W W

(Fig. 2)
B W W W
B B W B
B W B B
W W W B

where the original square is $\begin{matrix}B&W\\B&B\end{matrix}$.

I shall now proceed to show (i) how a square of 8 in each side can be formed subject to the condition that, on comparing any two rows, &c.; firstly, with the number of black and white squares the same; secondly, when it is not the same; and (ii) to show that this can be extended by induction to the case of a chessboard containing 2^i in each side.

(i)$_\alpha$. To prove it is true for a square of 8 when the number of black and of white is to be the same. Let us take three squares (Fig. 1) of four in each side to represent three whites; and a square with all the colours unlike those in Fig. 1 to represent a black, thus $\begin{matrix}A_1&B_1\\A_2&A_3\end{matrix}$ where B_1 is put for

W B W B
W W W W
W B B W
W W B B

Let these 3 whites be represented by A_1, A_2, A_3, and the black by B_1, as seen in the figure. Then, on comparing any row in $A_1 + A_2$, say the 3rd, with any row in $B_1 + A_3$, there will be 4 like and 4 unlike.

Firstly, let us take the 3rd row in $A_1 + A_2$ with the 3rd row in $B_1 + A_3$:

the 3rd in A_1 has all (*i. e.* 4) unlike the 3rd in B_1, by hypothesis;
the 3rd in A_2 has all like the 3rd in A_3;

therefore the 3rd in $A_1 + A_2$ has 4 like and 4 unlike the 3rd in $B_1 + A_3$.

Secondly, let us take any row in $B_1 + A_3$:

the 3rd in A_1 has 4 unlike the 3rd in B_1 which has 2 unlike the nth in B_1;
the 3rd in A_2 has 4 like the 3rd in A_3 which has 2 like the nth in A_3;

therefore the 3rd in $A_1 + A_2$ has 4 like the nth in $B_1 + A_3$,
also the 3rd in $A_1 + A_2$ has 4 like the nth in $A_1 + A_2$, by hypothesis.

Similarly the mth in $A_1 + A_2$ has 4 like the nth in $B_1 + A_3$.
and the mth in $A_1 + A_2$ has 4 like the mth in $A_1 + A_2$;

therefore the mth row has 4 like and 4 unlike in any column.

The proof is identical for horizontal columns. Therefore, on comparing any two columns, &c.

(i)$_\beta$. To show how to make the square of 8 in each side when the number of black and white is to be the same. The construction and proof of this are the same as before, merely using Fig. 2 for Fig. 1. We shall then get a

figure of the kind marked Fig. 3 below, whereas in the other case we get a figure similar to that marked Fig. 4.

B	W	W	W	B	W	W	W
B	B	W	B	B	B	W	B
B	W	B	B	B	W	B	B
W	W	W	B	W	W	W	B
B	W	W	W	W	B	B	B
B	B	W	B	W	W	B	W
B	W	B	B	W	B	W	W
W	W	W	B	B	B	B	W

FIG. 3.

B	W	B	W	B	W	B	W
B	B	B	B	B	B	B	B
B	W	W	B	B	W	W	B
B	B	W	W	B	B	W	W
B	W	B	W	W	B	W	B
B	B	B	B	W	W	W	W
B	W	W	B	W	B	B	W
B	B	W	W	W	W	B	B

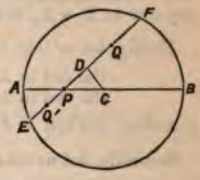
FIG. 4.

(ii.) To show how this can be extended to the case of a chessboard containing 2^i squares in each direction. It can be shown in a similar manner to the proof of the square of 8 in each side, that if it holds for a square containing 2^n, it holds for 2^{n+1} in each side. But it holds for 4 and 8, therefore it holds universally for a board containing 2^i in each side.

2407. (Proposed by the EDITOR.)—Show (1) that the average distance of all the pairs of points that can be taken within a circle, in parts of the radius, is $\frac{128}{45\pi}$; and (2) that the same average for a sphere is $\frac{36}{35}$.

Solution by STEPHEN WATSON.

1. Let C be the centre of the given circle, P and Q any two points within it, and AB a diameter through P. Put $CA=a$, $CP=x$, $PQ=y$, and $\angle QPB=\phi$. Then an element of the circle at P $= 2\pi x\,dx$, and at Q $= y\,d\phi\,dy$; and the number of positions of P and Q is π^2a^4. The limits of x are 0 and a, of ϕ are 0 and $\frac{1}{2}\pi$ and the result doubled, and of y are 0 and $(a^2-x^2\sin^2\phi)^{\frac{1}{2}}+x\cos\phi$, when Q lies on PF; but when Q lies on PE, as at Q′, and $PQ'=y$, the limits are 0 and $\left\{(a^2-x^2\sin^2\phi)^{\frac{1}{2}}-x\cos\phi\right\}$; thus the required average is



$$\frac{4}{\pi a^4}\int_0^{\frac{1}{2}\pi}d\phi\int_0^a x\,dx\int y^2\,dy,$$

which, when integrated for y in the manner stated above, becomes

$$=\frac{4}{\pi a^4}\int_0^{\frac{1}{2}\pi}d\phi\int_0^a\left\{\tfrac{2}{3}(a^2-x^2\sin^2\phi)^{\frac{3}{2}}+2x^2\cos^2\phi\,(a^2-x^2\sin^2\phi)^{\frac{1}{2}}\right\}x\,dx$$

$$=\frac{8a}{15\pi}\int_0^{\frac{1}{2}\pi}\frac{1+\cos^2\phi-6\cos^5\phi+4\cos^7\phi}{\sin^4\phi}\,d\phi=\frac{128a}{45\pi}.$$

2. In this case an element at $P = 4\pi x^2 dx$, and at $Q = 2\pi y \sin\phi\, y\, d\phi\, dy$; and the total positions of P and Q is $\frac{1}{3}\pi^2 a^6$. The limits are as before, except that the integral for ϕ must not be doubled; hence the average is

$$\frac{9}{2a^6}\int_0^{\frac{1}{2}\pi} \sin\phi\, d\phi \int_0^a x^2\, dx \int y^3\, dy,$$

which, integrated for y as in the preceding case, becomes

$$= \frac{9}{4a^6}\int_0^{\frac{1}{2}\pi} \sin\phi\, d\phi \int_0^a \left\{(a^2 - x^2\sin^2\phi)^2 + 6x^2\cos^2\phi\,(a^2 - x^2\sin^2\phi) + x^4\cos^4\phi\right\} \times x^2\, dx$$

$$= \frac{6a}{35}\int_0^{\frac{1}{2}\pi} (1 + 6\cos^2\phi + 15\cos^4\phi)\sin\phi\, d\phi = \frac{36a}{35}.$$

2702. (Proposed by W. S. McCay, B.A.)—The locus, in areal coordinates, of the centre of an ellipse, of constant area (πn^2), circumscribing a triangle, is $(x+y+z)(x+y-z)(x-y+z)(-x+y+z) = \dfrac{4x^2y^2z^2}{n^4}$.

I. *Solution by the* Rev. J. L. Kitchin, M.A.

If the conic be given in areal coordinates by the equation $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$, the centre (x, y, z) will be given by the equations

$$\frac{x}{l(l-m-n)} = \frac{y}{m(m-n-l)} = \frac{z}{n(n-l-m)} = \frac{2\Delta}{l^2+m^2+n^2-2mn-2nl-2lm} \quad \ldots(1).$$

Now the area (Whitworth's *Modern Geometry*, p. 281) $= \dfrac{2\pi\Delta H}{(-K)^{\frac{3}{2}}} = \pi n^2$,

where $K = \frac{1}{4}(l^2+m^2+n^2-2mn-2nl-2lm)$, and $H = \frac{1}{4}lmn$.

From the values of x, y, z, given by (1), we readily obtain

$$(x+y+z)(x+y-z)(x-y+z)(-x+y+z) = \frac{\Delta^4(l-m-n)^2(m-n-l)^2(n-l-m)^2}{-4K^3};$$

also $$\left(\frac{\Delta}{2K}\right)^6 = \frac{x^2y^2z^2}{l^2m^2n^2(l-m-n)^2(m-n-l)^2(n-l-m)^2};$$

$\therefore (x+y+z)(x+y-z)(x-y+z)(-x+y+z)\dfrac{\Delta^6}{(2K)^6} = \dfrac{-\Delta^4x^2y^2z^2}{64H^2K^3}$;

$\therefore (x+y+z)(x+y-z)(x-y+z)(-x+y+z) = x^2y^2z^2\left(\dfrac{-K^3}{\Delta^2H^2}\right) = \dfrac{4x^2y^2z^2}{n^4}$.

II. *Solution by the* Proposer.

Using the notation of the eccentric angle, the central perpendicular on

$\frac{x}{a}\cos\frac{\beta+\gamma}{2}+\frac{y}{b}\sin\frac{\beta+\gamma}{2}=\cos\frac{\beta-\gamma}{2}$ is $X=\frac{ab}{b'}\cos\frac{\beta-\gamma}{2}$, where b' is the parallel semi-diameter. But calling d, e, f the sides of the given triangle, we know (Salmon's *Conics*, p. 205) that $d=2b'\sin\frac{1}{2}(\beta-\gamma)$, therefore

$$\frac{dX}{ab}=\sin(\beta-\gamma);\quad\text{similarly}\quad\frac{eY}{ab}=\sin(\gamma-\alpha),\quad\text{and}\quad\frac{fZ}{ab}=\sin(\alpha-\beta).$$

But if $\theta+\theta'+\theta''=0$, it is easy to show that

$$(\sin\theta+\sin\theta'+\sin\theta'')(\sin\theta+\sin\theta'-\sin\theta'')(\sin\theta-\sin\theta'+\sin\theta'')\times(-\sin\theta+\sin\theta'+\sin\theta'')=4\sin^2\theta\sin^2\theta'\sin^2\theta''.$$

Hence we obtain at once the above equation for the locus.

2631. (Proposed by W. H. LAVERTY, B.A.)—A conic F has double contact with one of a set of conics having double contact with one another. Show that the chords of intersection of F with each of the set will intersect on the common chord of contact of the whole set.

Solution by the PROPOSER.

Project the set into a series of concentric circles, and the conic into a conic having double contact with one of the circles. Therefore, the common centre of the circles will lie on one of the principal axes of the conic; consequently, all the chords of intersection will be parallel, *i. e.*, will intersect on the line at infinity which is the common chord of contact of the circles.

2267. (Proposed by W. H. LAVERTY.)—If ∇ represent the determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ and if a', b', c', f', g', h', be the minors which respectively are the coefficients of a, b, c, f, g, h, in the expansion of ∇: then show that the coordinates (x, y) of the foci of $(abcfgh)(xy1)^2=0$, are given by

$$2c'x-2g'=\nabla^{\frac{1}{2}}\left\{(a-b+2hi)^{\frac{1}{2}}+(a-b-2hi)^{\frac{1}{2}}\right\},$$

and

$$2c'yi-2f'i=\nabla^{\frac{1}{2}}\left\{(a-b+2hi)^{\frac{1}{2}}-(a-b-2hi)^{\frac{1}{2}}\right\},$$

where i represents $\sqrt{-1}$. Also show that $(c'x-g')(c'y-f')=\Delta h$ is the equation to an hyperbola concentric with the given conic and passing through its four foci.

Solution by the PROPOSER.

The foci are the intersections of the pairs of tangents drawn from the circular points at infinity to the conic. Now the coordinates of these circular points are

$$\frac{x}{i}=\frac{y}{1}=p,\quad\text{and}\quad\frac{x}{-i}=\frac{y}{1}=q;$$

where p and q are infinite; and therefore in any equation involving no other infinite quantities, we must neglect all but the higher powers of p or q, and then divide through by that power. Remembering this, we get for the pair of tangents from the first circular point

$$(ax^2+by^2+2hxy+2gx+2fy+c)\ (-a+2hi+b) = (iu_x+u_y)^2,$$

whence, reducing and solving as a quadratic in $(x-yi)$, we find

$$(ab-h^2)\ (x-yi)-(g'-if') = \pm\nabla^{\frac{1}{2}}(a-b-2hi)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (1).$$

Similarly for the other two we get

$$(ab-h^2)\ (x+yi)-(g'+if') = \pm\nabla^{\frac{1}{2}}(a-b+2hi)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (2).$$

Adding and subtracting these alternately, we get the two required equations.

Also $$(2)^2-(1)^2 = 4i\,(c'x-g')\ (c'y-f') = 4ih\nabla,$$

another form of which will be $c'xy-g'y-f'x+h' = 0$.

The expressions for the coordinates of the foci simplify very much when $h = 0$.

2624. (Proposed by M. JENKINS, B.A.)—If ABC, A″B″C″ be two triangles, and T any point such that the area of the triangle $(\alpha\beta\gamma)$ formed by the intersections (α, β, γ) of TA″, TB″, TC″ respectively with BC, CA, AB, is constant; then the locus of T is a cubic having only one arbitrary parameter.

I. *Solution by the* PROPOSER.

For if h be the given constant and (C_h) denote the required locus, (C_h) passes through the intersections of (C_0) and (C_∞). Now if α', β', γ' be the intersections of BC, B″C″, &c., (C_0) is evidently a locus passing through A, B, C, A″, B″, C″, α', β', γ', and it may be readily shown that the locus cannot cut the straight line BC in more points than B, C, and α'; therefore C_0 is a cubic; and it will *in general* be defined by the nine points named. Also (C_∞) is the three straight lines through A″, B″, C″, drawn parallel to BC, CA, AB, respectively, whence follows the proposition given above.

When A″, B″, C″ are in the same straight line, α', β', γ' are also in the straight line A″B″C″, and the nine points named above are not sufficient to determine the cubic (C_0). But if D, E, F be the intersections of BC″, CB″, &c., then, in the general case, D, E, F are also points on (C_0). Hence, in the particular case, (C_0) breaks up into the straight line A″B″C″ and the conic S_0 (ABCDEF). These six points must lie on a conic, because (AF, CD), (BD, AE), (CE, BF) are three collinear points. If we now fix the positions of the points A″, B″, C″ on the straight line A″B″C″, so that they shall be in involution with α', β', γ', we shall find that EF passes through α', &c., and the conic through P, Q, the foci of the involution; and if A′, B′, C′ be the poles of BC, CA, AB, respectively, that A″A′, B″B′, C″C′ pass through (R) the pole of A″B″C″PQ, and thus obtain a generalization of Quest. 2395.

(C_h) does not necessarily break up because (C_0) does; but if A″B″C″ be the line at infinity, it gives an indeterminate area, and forms part of the

(C_0), (C_∞), and (C_h) loci. Hence, in this case, the (C_h) cubic breaks up into the line at infinity and a conic. The conic passes through the six points of intersection with the original conic of three straight lines, which are parallel to BC, CA, AB, respectively, and are at an infinite distance; that is, the conic is similar to and coaxial with the original conic.

II. *Solution by* J. J. WALKER, M.A.

Let ABC be taken as the triangle of reference, and the trilinear coordinates of T, A″, B″, C″ be (α, β, γ), $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ respectively; then the coordinates of the intersections of TA″ with BC, of TB″ with AC, and of TC″ with AB, will be

$$\begin{matrix} 0, & \dfrac{S(\alpha\beta_1-\beta\alpha_1)}{S_1\alpha-S\alpha_1}, & \dfrac{S(\alpha\gamma_1-\gamma\alpha_1)}{S_1\alpha-S\alpha_1}, \\ \dfrac{S(\beta\alpha_2-\alpha\beta_2)}{S_2\beta-S\beta_2}, & 0, & \dfrac{S(\beta\gamma_2-\gamma\beta_2)}{S_2\beta-S\beta_2}, \\ \dfrac{S(\gamma\alpha_3-\alpha\gamma_3)}{S_3\gamma-S\gamma_3}, & \dfrac{S(\gamma\beta_3-\beta\gamma_3)}{S_3\gamma-S\gamma_3}, & 0, \end{matrix}$$

where $2S = a\alpha+b\beta+c\gamma$, $2S_1 = a\alpha_1+b\beta_1+c\gamma_1$, $2S_2 = a\alpha_2+\ldots$ $2S_3 = a\alpha_3+\ldots$

The area of the triangle formed by lines joining these three points of intersection is

$$\frac{abc\,S}{8}\,\frac{(\gamma\beta_3-\beta\gamma_3)(\alpha\gamma_1-\gamma\alpha_1)(\beta\alpha_2-\alpha\beta_2)+(\beta\gamma_2-\gamma\beta_2)(\gamma\alpha_3-\alpha\gamma_3)(\alpha\beta_1-\beta\alpha_1)}{(S_1\alpha-S\alpha_1)(S_2\beta-S\beta_2)(S_3\gamma-S\gamma_3)},$$

and the numerator of the latter factor I find can be thrown into the form $[(\beta_3\gamma_2)\ldots$ meaning $(\beta_3\gamma_2-\beta_2\gamma_3)\ldots]$

$$\Sigma = \alpha_1\left\{(\beta_3\gamma_2)\alpha+(\gamma_3\alpha_2)\beta+(\alpha_3\beta_2)\gamma\right\}\beta\gamma+\beta_2\left\{(\gamma_3\beta_1)\alpha+(\alpha_3\gamma_1)\beta+(\beta_3\alpha_1)\gamma\right\}\gamma\alpha$$
$$+\gamma_3\left\{(\beta_2\gamma_1)\alpha+(\gamma_2\alpha_1)\beta+(\alpha_2\beta_1)\gamma\right\}\alpha\beta+\left\{\alpha_1(\beta_2\gamma_3)+\beta_1(\gamma_3\alpha_2)+\gamma_1(\alpha_2\beta_3)\right\}\alpha\beta\gamma.$$

If the area of the triangle in question be constant (say equal to $\frac{1}{8}S'$) the locus of T will be the cubic

$$abc\,S\Sigma-S'(S_1\alpha-S\alpha_1)(S_2\beta-S\beta_2)(S_3\gamma-S\gamma_3) = 0 \ldots\ldots\ldots (1),$$

which locus (it may easily be verified) passes through A″, B″, C″.

Now if the three points A″, B″, C″ lie on a right line $l\alpha+m\beta+n\gamma = 0$, the coefficient of $\alpha\beta\gamma$ (as far as it appears explicitly) in Σ must vanish, and the remaining terms of Σ will have $l\alpha+m\beta+n\gamma$ as a common factor; in fact, Σ may then be written

$$(l\alpha+m\beta+n\gamma)\left\{\alpha_1(\beta_3\gamma_2)\frac{\beta\gamma}{l}+\beta_2(\alpha_3\gamma_1)\frac{\gamma\alpha}{m}+\gamma_3(\alpha_2\beta_1)\frac{\alpha\beta}{n}\right\};$$

and it at once appears that, for the value $S' = 0$, the locus of T breaks up into the right line $l\alpha+m\beta+n\gamma = 0$ and the conic $\alpha_1(\beta_3\gamma_2)\frac{\beta\gamma}{l}+\ldots = 0$.

Finally, let the line $l\alpha+m\beta+n\gamma = 0$ become the line at infinity, and the three points A″, B″, C″ be determined on it as the poles of those diameters of any conic $L\beta\gamma+M\gamma\alpha+N\alpha\beta = 0$, circumscribing the triangle ABC, which are parallel to BC, AC, AB respectively, the area S′ being again finite. It will

now be found that the locus of T is the line at infinity, and the conic

$$abc\left\{\frac{(\beta_3\gamma_2)}{\beta_2\gamma_3}\frac{\beta\gamma}{a}+\frac{(\alpha_3\gamma_1)}{\alpha_1\gamma_3}\frac{\gamma\alpha}{b}+\frac{(\alpha_2\beta_1)}{\alpha_1\beta_2}\frac{\alpha\beta}{c}\right\}-S'S^2=0\ldots\ldots(2),$$

where S is the area of the triangle ABC, since finite points only are considered. By forming the equations of the lines joining the points A, B, C with A″, B″, C″, it will be found that

$$\frac{\beta_1}{\alpha_1}=-\frac{aL+bM-cN}{2bL},\quad \frac{\alpha_2}{\beta_2}=-\frac{aL+bM-cN}{2aM},\quad \frac{\gamma_1}{\alpha_1}=\ldots;$$

and substituting these expressions in (2), the equation becomes

$$(a^2L^2+b^2M^2+c^2N^2-2bcMN-2caNL-2abLM)(L\beta\gamma+M\gamma\alpha+N\alpha\beta)$$
$$-4LMNS'S^2=0,$$

a conic similar to and coaxal with $L\beta\gamma+M\gamma\alpha+N\alpha\beta=0$ [Quest. 2464].

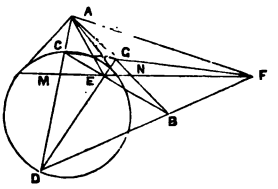
2383. (Proposed by W. K. Clifford, B.A.)—A and B are fixed points with regard to a conic, ACD a variable straight line passing through A and cutting the curve in C, D. The polar of A meets BC, BD in E, F; show that DE and CF meet in a fixed point G, and that ABG is a straight line.

Solution by J. Dale; W. H. Laverty, B.A.; *and others.*

Drawing the figure as directed, let M, N be the points in which the polar of A is cut by ACD and AB. Then

(E . ACMD) = (F . ACMD) = (F . ADMC);

therefore ED and FC intersect in the fixed line AB, and since the point of intersection G forms a harmonic range with the fixed points A, N, B, it is also fixed in position.



2551. (Proposed by Chief Justice Cockle, F.R.S.)—Given the partial differential equation $\frac{d^2z}{dy^2}-\alpha^2\frac{d^2z}{dx^2}+\beta\frac{dz}{dy}+\gamma z=0\ldots\ldots\ldots\ldots\ldots(I)$,

wherein α, β, and γ are functions of x only. Show that when the condition

$$\gamma-\frac{1}{4}\left(\frac{d\alpha}{dx}+\beta\right)^2-\frac{\alpha^2}{2}\frac{d}{dx}\left\{\frac{1}{\alpha}\left(\frac{d\alpha}{dx}+\beta\right)\right\}=0\ldots\ldots\ldots(II)$$

is satisfied, then in general $z=\sqrt{\alpha}\,\epsilon^{\int\frac{\beta}{2\alpha}dx}\,\phi\left(y+\int\frac{dx}{\alpha}\right)$,

where ϕ is an arbitrary function. Show also that, when β is constant and (11) satisfied, then the integral of (I) is

$$x = \sqrt{\alpha}\, \epsilon^{\int \frac{\beta}{2\alpha} dx} \phi\left(y + \int \frac{dx}{\alpha}\right) + k\epsilon^{-\int \frac{\beta}{2\alpha} dx} \psi\left(y - \int \frac{dx}{\alpha}\right),$$

where ψ is another arbitrary function, and the arbitrary multiplier k is only inserted for convenience.

Solution by the PROPOSER.

From the equation $z = A\phi(y + B)$ .....(1),

where ϕ is arbitrary and A and B are functions of x only, there results

$$\frac{d^3z}{dy^3} - \frac{1}{B'^2}\frac{d^3z}{dx^2} + \left(\frac{2A'B' + AB''}{AB'^2}\right)\frac{dz}{dy} + \frac{A''}{AB'^2} z = 0 \text{ .....(2)},$$

which we may represent by $\frac{d^3z}{dy^3} - \alpha^2\frac{d^3z}{dx^3} + \beta\frac{dz}{dy} + \gamma z = 0$ .....(3);

so that $\frac{1}{B'} = \alpha, \quad \frac{2A'B' + AB''}{AB'^2} = \beta, \quad \frac{A''}{AB'^2} = \gamma$ .....(4, 5, 6).

Eliminating B' by means of (4) from (5) and (6), we have

$$-\alpha'A + 2\alpha A' = \beta A, \quad \text{and} \quad \alpha^2 A'' = \gamma A \text{ .....(7, 8)},$$

the accents throughout denoting differentiations with respect to x.

From (7) we deduce by differentiation

$$-\alpha''A + \alpha'A' + 2A''\alpha = \beta A' + \beta' A \text{ .....(9)}.$$

Divide (9) by A, and by means of the sinister forms of (7) and (8) eliminate $\frac{A'}{A}$ and $\frac{A''}{A}$ from the quotient. We arrive at

$$-\alpha'' + \frac{\alpha'}{2\alpha}(\alpha' + \beta) + \frac{2\gamma}{\alpha} = \frac{\beta}{2\alpha}(\alpha' + \beta) + \beta' \text{ .....(10)},$$

or $$\frac{2\gamma}{\alpha} + \frac{(\alpha' - \beta)(\alpha' + \beta)}{2\alpha} - \alpha'' - \beta' = 0 \text{ .....(11)},$$

which reduces to $$\gamma + \tfrac{1}{4}(\alpha'^2 - \beta^2) - \tfrac{1}{2}(\alpha\alpha'' + \alpha\beta') = 0 \text{ .....(12)},$$

whence $$\gamma - \tfrac{1}{4}(\alpha' + \beta)^2 + \tfrac{1}{2}(\alpha'^2 + \beta\alpha') - \tfrac{1}{2}(\alpha\alpha'' + \alpha\beta') = 0,$$

or lastly $$\gamma - \tfrac{1}{4}(\alpha' + \beta)^2 - \frac{\alpha^2}{2}\frac{d}{dx}\left(\frac{\alpha' + \beta}{\alpha}\right) = 0 \text{ .....(13)}.$$

Whenever the condition (13) is fulfilled, (3) is satisfied by (1). Moreover B', and consequently B, is obtained from (4), and we also have, by (5),

$$2\frac{A'}{A} + \frac{B''}{B'} = \beta B' \text{ .....(14)},$$

whence $A^2B' = \epsilon^{\int \beta B' dx}$, or, by (4), $A^2 = \alpha\epsilon^{\int \frac{\beta}{\alpha} dx}$ ..... (15, 16).

Next, let $$z = A\phi(y + B) + C\psi(y - B) \text{ .....(17)},$$

the functions ϕ and ψ being arbitrary, and A, B, and C being functions of x only. Also let $\frac{d^n\phi(y+B)}{d(y+B)^n} = \phi_n$, $\frac{d^n\psi(y-B)}{d(y-B)^n} = \psi_n$ .....(18, 19);

then we have $\frac{dz}{dy} = A\phi_1 + C\psi_1$, $\frac{d^2z}{dy^2} = A\phi_2 + C\psi_2$ .....(20, 21),

$$\frac{dz}{dx} = A\phi_1 B' + A'\phi - C\psi_1 B' + C'\psi \text{ .....(22),}$$

$$\frac{d^2z}{dx^2} = \left.\begin{array}{l} A\phi_2 B'^2 + (2A'B' + AB'')\phi_1 + A''\phi \\ + C\phi_2 B'^2 - (2C'B' + CB'')\psi_1 + C''\psi \end{array}\right\} \text{ .....(23).}$$

Hence (17) will give rise to an equation of the form of (2) or (3), provided that

$$-\left(\frac{2C'B' + CB''}{CB'^2}\right) = \frac{2A'B' + AB''}{AB'^2} = \beta, \text{ and } \frac{C''}{CB'^2} = \frac{A''}{AB'^2} = \gamma \text{ ... (24, 25).}$$

But one branch of (24) gives $\frac{2C'B' + CB''}{CB'^2} + \frac{2A'B' + AB''}{AB'^2} = 0$ .....(26),

which, multiplied into B' and reduced, leads, on division by (2), to

$\frac{C'}{C} + \frac{B''}{B'} + \frac{A'}{A} = 0$, whence, by integration, $ACB' = k$ ....(27, 28),

k being an arbitrary constant. Moreover one branch of (25) gives

$CA'' - AC'' = 0$, whence, by integration, $CA' - AC' = K$ ....(29, 30),

where K is another arbitrary constant. We have still, it will be remembered, from the second branch of (24) $\frac{2A'}{A} + \frac{B''}{B'} = \beta B'$ .....(14).

Hence, subtracting (27) from (14), $\frac{A'}{A} - \frac{C'}{C} = \beta B'$ .....(31).

But, dividing (30) by AC, we have $\frac{A'}{A} - \frac{C'}{C} = \frac{K}{AC}$ .....(32);

consequently, by (31) and (32), $\beta B' = \frac{K}{AC}$ .....(33);

but, by (28), $B' = \frac{k}{AC}$ .....(34);

hence, by (33) and (34) combined, $\beta = \frac{K}{k}$ .....(35).

The inference from this is, that when β is constant and (13) is satisfied, (3) has (17) for the form of its integral. Moreover, by (16),

$$A = \sqrt{a}\, \epsilon^{\int \frac{\beta}{2a} dx} \text{ .....(36);}$$

consequently by (34) and (4) $C = k\sqrt{a}\, \epsilon^{-\int \frac{\beta}{2a} dx}$ .....(37);

so that, when (13) is satisfied and β constant, the integral of (3) is

$$z = \sqrt{a}\, \epsilon^{\int \frac{\beta}{2a} dx} \phi\left(y + \int \frac{dx}{a}\right) + k\sqrt{a}\, \epsilon^{-\int \frac{\beta}{2a} dx} \psi\left(y - \int \frac{dx}{a}\right) \text{ ....(38).}$$

It may be well to test these results by an example. Given

$$\frac{d^2z}{dy^2} - \frac{d^2z}{dx^2} + 2\frac{dz}{dy} + z = 0.$$

Here $\alpha=1$, $\beta=2$, $\gamma=1$, the condition (13) is satisfied, and the solution derived from (38) is $z = \epsilon^x\phi(y+x) + k\epsilon^{-x}\psi(y-x)$,

which is verified as follows :—

$$\frac{dz}{dy} = \epsilon^x\phi_1 + k\epsilon^{-x}\psi_1,$$

$$\frac{d^2z}{dy^2} = \epsilon^x\phi_2 + k\epsilon^{-x}\psi_2,$$

$$\frac{dz}{dx} = \epsilon^x\phi_1 + \epsilon^x\phi - k\epsilon^{-x}\psi_1 - k\epsilon^{-x}\psi,$$

$$\frac{d^2z}{dx^2} = \epsilon^x\phi_2 + 2\epsilon^x\phi_1 + \epsilon^x\phi + k\epsilon^{-x}\psi_2 + 2k\epsilon^{-x}\psi_1 + k\epsilon^{-x}\psi,$$

whence $$\frac{d^2z}{dy^2} - \frac{d^2z}{dx^2} = -2(\epsilon^x\phi_1 + k\epsilon^{-x}\psi_1) - (\epsilon^x\phi + k\epsilon^{-x}\psi),$$

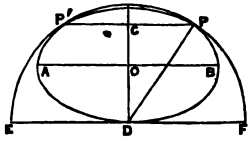
or $$\frac{d^2z}{dy^2} - \frac{d^2z}{dx^2} + 2\frac{dz}{dy} + z = 0,$$

as it ought.

2577. (Proposed by the EDITOR.)—Show that the average area of all the ellipses that can be drawn touching a semicircle and its diameter symmetrically, in parts of the area of the ellipse, is $\frac{2}{15}(2+\sqrt{2})$, or ·45523..., if (1) all possible positions of the centre of the ellipse be supposed equally probable, and $(2-\sqrt{2})$, or ·58579..., if (2) all distances of the chord of contact from the centre of the semicircle be supposed equally probable.

Solution by S. WATSON; *the* REV. J. WOLSTENHOLME, M.A.; *and others.*

1. It is plain the ellipse must always touch the diameter EF of the semicircle in its middle point D. Let O be the centre of the ellipse, and PCP′ the chord of contact with the semicircle. Put ED=DP=r, DC=z; and let $(2\alpha, 2\beta)$ be the axes of the ellipse, and (x, y) the coordinates of P relative to these axes. Then we have



$$r = \text{maximum value of } \left\{(\beta+y)^2 + x^2\right\}^{\frac{1}{2}}$$

$$\cdots = \text{maximum value of } \left\{(\beta+y)^2 + \alpha^2 - \frac{\alpha^2y^2}{\beta^2}\right\}^{\frac{1}{2}} = \frac{\alpha^2}{(\alpha^2-\beta^2)^{\frac{1}{2}}};$$

therefore $\beta^2 = a^2 - \frac{a^4}{r^2}$, and $\beta\, d\beta = \left(a - \frac{2a^3}{r^2}\right) da$.

Now, in (1), β may, with equal probability, take any value from 0 to $\frac{1}{2}r$; therefore $\frac{1}{2}r$ is the measure of the number of ellipses. Also the limits of a are r and $\frac{1}{2}r\sqrt{2}$; hence, in this case, the average area of the ellipse in parts of the semicircle is

$$\frac{2}{r}\int_0^{\frac{1}{2}r} \frac{a\beta d\beta}{\frac{1}{2}r^2} = \frac{4}{r^3}\int_r^{\frac{1}{2}r\sqrt{2}} \left(1 - \frac{2a^2}{r^2}\right) a^2 da = \frac{2}{15}(2+\sqrt{2}) = \cdot 45523\ldots$$

2. The value of y belonging to the maximum in (1) is

$$y = \frac{\beta^2}{a^2 - \beta^2}, \quad \text{therefore } z = \beta + y = \frac{a^2\beta}{a^2-\beta^2} = \frac{r(r^2-a^2)^{\frac{1}{2}}}{a}.$$

Also, since z may in this case, with equal probability, take any value from 0 to r, the measure of the number of ellipses is r; hence the average area of the ellipse in parts of the semicircle is

$$\frac{1}{r}\int_0^r \frac{a\beta dz}{\frac{1}{2}r^2} = -\frac{2}{r}\int_r^{\frac{1}{2}r\sqrt{2}} da = 2 - \sqrt{2} = \cdot 58579\ldots$$

[Referred to DF, DC, as axes of coordinates, the equation to a conic having double contact along the chord PP′ (or $y - z = 0$) with the circle $x^2 + y^2 - r^2 = 0$, that is the given semicircle APP′E, is

$$x^2 + y^2 - r^2 + k(y-z)^2 = 0,$$

and determining k so that the conic may touch DF at D (*i. e.*, the axis of x at the origin), we find $k = \frac{r^2}{z^2}$; thus the equation to the ellipse DPP′ is

$$x^2 + \left(1 + \frac{r^2}{z^2}\right) y^2 - \frac{2r^2}{z} y = 0, \text{ or } \left(\frac{r^2+z^2}{r^4}\right) x^2 + \frac{(r^2+z^2)^2}{r^4 z^2}\left(y - \frac{r^2 z}{r^2+z^2}\right)^2 = 1,$$

and the semiaxes are $a = \frac{r^2}{(r^2+z^2)^{\frac{1}{2}}}$, $\beta = \frac{r^2 z}{r^2+z^2}$;

whence we obtain $z = \frac{r}{a}(r^2-a^2)^{\frac{1}{2}}$, $\beta^2 = a^2 - \frac{a^4}{r^2}$,

as in the above solution. See also Hymers' *Conics*, p. 189.]

2738. (Proposed by R. Moon, M.A.)—The equation representing the transmission of sound parallel to the axis of a tube, when formed without reference to any previously assumed law of pressure, is $\frac{d^2y}{dt^2} + \frac{1}{D} \cdot \frac{dp}{dx} = 0$, where x is the ordinate of the point of rest of a particle, y the same ordinate at the time t, p the pressure at the same time, and D the density of equi-

Ebrium. Show that the equation is satisfied by the values of the pressure, velocity, and density derived from the three following equations, viz.

$$p = -\frac{a^2}{\rho} + \chi\left(v + \frac{a}{\rho}\right), \quad v + \frac{a}{\rho} = \phi\left(x + \frac{a-\chi'}{D}t\right),$$

$$\frac{1}{\rho} - f\left(v + \frac{a}{\rho}\right) = \psi\left(x - \frac{a}{D}t\right),$$

where a is an arbitrary constant; χ, ϕ, and ψ are arbitrary functions; and f is a function whose form is given by the equation

$$f(z) = \int \frac{dz}{2a - \chi'(z)}$$

Solution by the Proposer.

The first equation of the solution gives

$$\frac{dp}{dx} = \frac{a^2}{\rho^2}\frac{d\rho}{dx} + \left\{\frac{dv}{dx} - \frac{a}{\rho^2}\frac{d\rho}{dx}\right\} \cdot \chi';$$

hence the differential equation of motion becomes

$$0 = \frac{dv}{dt} + \frac{\chi'}{D} \cdot \frac{dv}{dx} + \frac{a}{\rho^2} \cdot \frac{a-\chi'}{D} \cdot \frac{d\rho}{dx} \quad \ldots\ldots\ldots\ldots (1).$$

The second equation of the solution gives us

$$\frac{d}{dx}\left(v + \frac{a}{\rho}\right) = \phi' \cdot \left\{1 - \frac{t}{D} \cdot \chi'' \cdot \frac{d}{dx}\left(v + \frac{a}{\rho}\right)\right\},$$

$$\frac{d}{dt}\left(v + \frac{a}{\rho}\right) = \phi' \cdot \left\{\frac{a-\chi'}{D} - \frac{t}{D} \cdot \chi'' \cdot \frac{d}{dt}\left(v + \frac{a}{\rho}\right)\right\};$$

whence we have $0 = \frac{d}{dt}\left(v + \frac{a}{\rho}\right) - \frac{a-\chi'}{D} \cdot \frac{d}{dx}\left(v + \frac{a}{\rho}\right),$

or

$$0 = \frac{dv}{dt} - \frac{a}{\rho^2}\frac{d\rho}{dt} - \frac{a-\chi'}{D}\left(\frac{dv}{dx} - \frac{a}{\rho^2}\frac{d\rho}{dx}\right) \quad \ldots\ldots\ldots\ldots (2).$$

But

$$\frac{dv}{dx} = \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{D}{\rho}\right) = -\frac{D}{\rho^2}\frac{d\rho}{dt};$$

and eliminating, by means of this, $\frac{d\rho}{dt}$ from (2), the latter becomes identical with (1). In like manner, the third equation of the solution gives

$$-\frac{1}{\rho^2}\frac{d\rho}{dx} + \frac{1}{\chi' - 2a} \cdot \frac{d}{dx}\left(v + \frac{a}{\rho}\right) = \psi',$$

$$-\frac{1}{\rho^2}\frac{d\rho}{dt} + \frac{1}{\chi' - 2a} \cdot \frac{d}{dt}\left(v + \frac{a}{\rho}\right) = -\frac{a}{D} \cdot \psi';$$

whence, eliminating ψ', we get

$$0 = -\frac{1}{\rho^2}\left(\frac{d\rho}{dt} + \frac{a}{D}\frac{d\rho}{dx}\right) + \frac{1}{\chi' - 2a} \cdot \left\{\frac{d}{dt}\left(v + \frac{a}{\rho}\right) + \frac{a}{D}\frac{d}{dx}\left(v + \frac{a}{\rho}\right)\right\}.$$

Eliminating $\frac{dp}{dt}$ as before, this last equation becomes identical with (1).

It thus appears that each of the two last equations of the proposed solution satisfies (1): and (1) is what the equation of motion becomes when the proposed value of p is substituted for it; *i.e.*, the relations between the pressure, velocity, density, ordinate, and time, above proposed, satisfy the equation of motion.

Moreover, not only is the equation of motion solved in the above manner with the requisite generality, but it is solved with a degree of generality greater than any one had supposed to be necessary, or conceived to be attainable.

On the other hand, no other solution of the complete equation of sound, possessing the necessary degree of generality, has ever been proposed; the most general hitherto obtained, that of Poisson, involving only one arbitrary function, which is universally admitted, and can be easily shown to be insufficient.

The cause of the failure of all attempts made heretofore towards obtaining the general solution, undoubtedly has been the vicious law of pressure, which, without proof or evidence, has been assumed in the discussion of the problem.

The absolutely groundless nature of this law, viz., the law expressed by the equation $p = a^2\rho$, may be shown in the clearest manner as follows:—

In any case of motion, the pressure must be capable of being expressed in terms of the ordinate and time. The same may be said of the density; the same of the velocity. Hence we are at liberty to assume

$$p = f_1(x, t), \quad \rho = f_2(x, t), \quad v = f_3(x, t).$$

From these three equations, we may eliminate two of the unknown quantities, as x and t; and so obtain a relation between the remaining three, viz., p, ρ and v. Solving the equation so obtained, we shall get $p = f(\rho, v)$. Now on what ground does the ordinary theory cut down this general equation to the particular equation $p = a^2\rho$? Simply on the ground that the latter formula expresses the pressure in the case of equilibrium.

How far this consideration is sufficient, may be seen as follows:—

In the case of equilibrium, we have

$$p = \text{funct. } x, \quad \rho = \text{funct. } x, \quad \text{whence } p = \text{funct. } \rho.$$

But we have no more reason to suppose that p is a function of ρ only in the case of motion, than we have to suppose that in the case of motion p is a function of x only; or ρ a function of x only; *i.e.*, we have no reason at all to suppose so, though the absurdity of such a supposition in the latter cases is palpable, while in the former case it is not immediately obvious.

With regard to the solution I have above put forth, I would observe, that it has been derived from an equation of motion formed irrespective of any theory as to the nature of the law of pressure; and it has been so derived, by assuming provisionally that the differential equation admits of *two* first integrals of the form given by Monge's method.

In order that that assumption may hold good, I have shown (see *Philosophical Magazine*, vol. xxxvi., p. 27) that *two* relations must obtain between p, ρ, and v, or their partial differential coefficients, both of which, by a remarkable coincidence, may be satisfied by a single relation between p, ρ, and v; viz., that given in the above solution. It may thus be seen that the fact of p being left disposable, instead of being arbitrarily determined beforehand, as in the received theory, is the cause of the success of the method which I have adopted; and that the assumption of the equation $p = a^2\rho$ is not only unwarrantable, but is positively mischievous.

drawing through A P′Q′, making an infinitesimal angle $d\theta$ with PQ, all the positions of second point B, which *give the direction of the line* AB *between the limits* θ *and* $\theta + d\theta$, are contained within the triangles PAP′, QAQ′; hence the number of such lines AB is measured by $\frac{1}{2}d\theta(\rho^2 + \rho'^2)$.

Hence, if dS be an element of the area at A, and if A now be not fixed, but *taken anywhere within* dS, the number of lines AB between the directions θ and $\theta + d\theta$ will be $\frac{1}{2}d\theta(\rho^2 + \rho'^2)\,dS$.

Consequently, if A be anywhere within the space between PQ and a parallel chord, distant dp from it (Fig. 2), the number of lines AB between θ and $\theta + d\theta$ will be

$$\tfrac{1}{2}d\theta\,dp\int_0^C \left\{\rho^2 + (C-\rho)^2\right\}d\rho = \tfrac{1}{3}C^3\,dp\,d\theta.$$

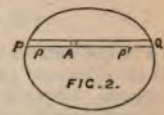


FIG. 2.

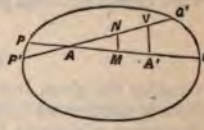
Now if A be *anywhere within the boundary*, the number of lines AB between the directions θ and $\theta + d\theta$, is $\frac{1}{3}d\theta\int C^3\,dp$, the chord C always having the direction θ. Hence, allowing θ also to vary, we have, for the total number of lines AB, $N = \frac{1}{3}\iint C^3\,dp\,d\theta$. But we know the total number to be Ω^2. Hence $\iint C^3\,dp\,d\theta = 3\Omega^2$.

NOTE.—Hence, if L be the perimeter of any convex area Ω, the average value of the cube of a chord drawn at random across it is $\frac{3\Omega^2}{L}$. The average value of the chord itself is $\frac{\pi\Omega}{L}$. Hence *the average length of a line drawn at random across a triangle is the same as that of one drawn at random across the inscribed circle, viz., a quarter of the circumference of that circle.* The same is true for any polygon circumscribed about a circle.

2680. (Proposed by Capt. CLARKE, R.E., F.R.S.)—If A and B be two points taken at random within a closed convex figure, and the line AB be produced to meet the boundary in P and Q, prove that the chance that the three segments PA, AB, BQ are such as would form a triangle is $\frac{1}{4}$.

Solution by M. W. CROFTON, F.R.S.

A solution of this question may be readily deduced from that of the preceding Question 2721. For let A be a fixed point within the boundary, and B a second point taken at random within it. Through A draw two chords PQ, P′Q′, whose directions are θ and $\theta + d\theta$. Confining our attention at present to lines AB between θ and $\theta + d\theta$, let us examine the favourable positions of B (that is, those which render it possible to form a triangle with the three segments of the line AB produced both ways to meet the boundary). Now if we bisect PQ in M, and take MA′ = AP, and draw MN, A′V perpendicular to PQ, it will be seen that all the favourable



and similarly for $\mathrm{E}\left(\frac{n}{p}\right)+$&c., $\mathrm{E}\left(\frac{m+n}{p}\right)+$&c.... ; whence

$$\begin{aligned}(p-1)\,\mathrm{N} = \quad & \delta\,(p^3-1)+\gamma\,(p^2-1)+\beta\,(p-1)\\ & -d\,(p^3-1)-c\,(p^2-1)-b\,(p-1)\\ & -d'(p^3-1)-c'(p^2-1)-c'(p-1)\\ = & \left\{m+n-\mathrm{S}\,(m+n)\right\}-\left\{m-\mathrm{S}\,(m)\right\}-\left\{n-\mathrm{S}\,(n)\right\}\\ = & \,\mathrm{S}\,(m)+\mathrm{S}\,(n)-\mathrm{S}\,(m+n) = (p-1)\,x,\end{aligned}$$

if x be the number of times of carrying for the sum $m+n$, or of borrowing for the difference $(m+n)-m$ or $(m+n)-n$; that is, $\mathrm{N}=x$, the required theorem. I remark that although the foregoing expression of the number N is a very elegant and ingenious one, yet the original form of N, as given at the end of (3), is the natural and proper expression of the number of times that the factor p occurs in the binomial coefficient $\frac{\Pi\,(m+n)}{\Pi\,(m)\,\Pi\,(n)}$.

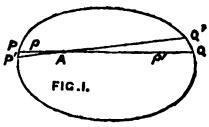
II. *Solution by the* PROPOSER.

If $\mathrm{I}\left(\frac{m}{p^r}\right)$ denote the greatest integer in $\frac{m}{p^r}$, the highest power of p in $\Pi\,(m)$ is $\mathrm{I}\left(\frac{m}{p}\right)+\mathrm{I}\left(\frac{m}{p^2}\right)+$&c.; and this may be shown to be equal to $\frac{m-\mathrm{S}_m}{p-1}$, where S_m denotes the sum of the digits of m expressed in the scale of p. Hence the required index is equal to $\frac{\mathrm{S}_m+\mathrm{S}_n-\mathrm{S}_{m+n}}{p-1}$, which evidently expresses the number of times it would be necessary to borrow in subtracting m (or n) from $m+n$; and this number can obviously be obtained by a simple inspection of the numbers m (or n) and $m+n$, expressed in the scale of p, without actually performing the subtraction.

2721. (Proposed by M. W. CROFTON, F.R.S.)—Given any convex area Ω, let C be the length of any chord drawn across the area, and let the position of C be defined by (p, θ), the perpendicular on it from a fixed pole inside Ω and the inclination of p to a fixed axis; then $\iint \mathrm{C}^3\,dp\,d\theta = 3\Omega^2$, the integration extending to all possible positions of C.

Solution by the PROPOSER.

Consider the given area as covered with points uniformly distributed: and let us examine the infinite system of straight lines obtained by joining every pair of points. Call the pair of points A, B; and first, let A be fixed. Through A (Fig. 1) draw a chord PQ inclined at an angle θ to some fixed direction. Put $\mathrm{AP}=\rho$, $\mathrm{AQ}=\rho'$, $\mathrm{PQ}=\mathrm{C}$. Then,



therefore $\frac{x}{a} = \cos\theta\cos 2\theta + 2\sin\theta\sin 2\theta = \cos^3\theta + 3\sin^2\theta\cos\theta,$

and $\frac{y}{b} = 2\cos\theta\sin 2\theta - \sin\theta\cos 2\theta = \sin^3\theta + 3\sin\theta\cos^2\theta;$

hence we have $\frac{x}{a} \pm \frac{y}{b} = (\cos\theta \pm \sin\theta)^3;$

and the equation of the envelope is

$$\left(\frac{x}{a} + \frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{x}{a} - \frac{y}{b}\right)^{\frac{2}{3}} = 2,$$

a curve having cusps (as in Fig. 2) on the equiconjugate diameters at the points $\frac{x^2}{a^2} = \frac{y^2}{b^2} = 2.$

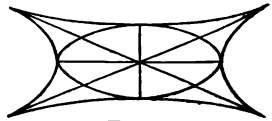


FIG. 2.

In the parabola $y^2 = 4ax$, the equation of the chord of intersection of the curve with the circle of curvature at $(am^2, 2am)$ is

$$y - 2am = -\frac{1}{m}(x - am^2), \quad \text{or} \quad 3am - y - \frac{x}{m} = 0,$$

whose envelope is $y^2 + 12ax = 0.$

II. *Solution by the* PROPOSER.

If the three pairs of lines through four points meet in X, Y, Z, then a line in the plane is touched by two conics through the four points in two points, harmonic conjugates to the conics of the system, including the three pairs of lines through them. The points are inverse to the triangle XYZ, their coordinates being connected by equations of the form $axx' = byy' = czz'$. The inverse to a conic of the system is easily found by taking on a tangent to the conic the harmonic conjugate to any pair of lines through the four points. The equation to the conic being of the form $\alpha x^2 + \beta y^2 + \gamma z^2 = 0$, the inverse curve is of the form $\alpha' x^2 + \beta' y^2 + \gamma' z^2 = 0$. A simple case of this is the problem (1).

If a conic is inscribed in four lines, the envelope of the line which, with a tangent to the conic, forms a pair of lines conjugate to the quadrilinear, is the reciprocal of the last mentioned locus. If the four lines intersect in the foci of the curve, the envelope is the evolute of the curve, and a full discussion of its properties is given by Prof. CREMONA in the *Messenger of Mathematics*, Vol. III., p. 88. It will readily be seen that, in (2), the circumscribing rectangle parallel to the axes forms the quadrilinear.

2733. (Proposed by the EDITOR.)—Show that the average area of an infinite series of triangles, equably distributed over the surface of an ellipse so as to have each one side parallel to a given line, in parts of the area of the ellipse, is $\frac{256}{525\pi^2}$.

I. *Solution by* THOMAS SAVAGE, M.A.

1. Imagine a series of chords of a *circle* drawn at very small equal distances parallel to a given line. Then we may consider the area of the circle to be filled with points uniformly distributed along these chords, the number of points in each chord being of course proportional to the length of the chord.

Let PQR be the triangle. The number of triangles in which P, Q fall on a chord at a distance x from the centre, is proportional to the square of the length of the chord, or to r^2-x^2, if r be the radius. The number of triangles in which R falls on a chord at a distance y from the centre, is proportional to $(r^2-y^2)^{\frac{1}{2}}$, and the average area of the triangles thus formed is

$$\tfrac{1}{2}\,(\text{average length of PQ})\,(y \curvearrowleft x).$$

Now, the average length of PQ is $\frac{2}{3}(r^2-x^2)^{\frac{1}{2}}$ as is easily seen. Hence, putting (Δ) for the average area of the triangle, PQR, we have

$$(\Delta) = \frac{\iint (r^2-x^2)\,(r^2-y^2)^{\frac{1}{2}}\,\tfrac{1}{2}\,(r^2-x^2)^{\frac{1}{2}}\,(y \curvearrowleft x)\,dx\,dy}{\iint (r^2-x^2)\,(r^2-y^2)^{\frac{1}{2}}\,dx\,dy},$$

where $y \curvearrowleft x$ is positive throughout the integration. Put $x = r\cos\alpha$, $y = r\cos\beta$, and the integrals in the value of (Δ) reduce to

$$\frac{\frac{r^2}{3}\cdot\int_0^{\pi}\sin^4\alpha\left\{\int_0^{\alpha}\sin^2\beta\,(\cos\beta-\cos\alpha)\,d\beta+\int_{\alpha}^{\pi}\sin^2\beta\,(\cos\alpha-\cos\beta)\,d\beta\right\}d\alpha}{\int_0^{\pi}\int_0^{\pi}\sin^3\alpha\,\sin^2\beta\,d\alpha\,d\beta}$$

Now $\displaystyle\int_0^{\alpha}\sin^2\beta\cos\beta\,d\beta = \frac{\sin^3\alpha}{3}$, $\displaystyle\int_{\alpha}^{\pi}\sin^2\beta\cos\beta\,d\beta = -\frac{\sin^3\alpha}{3}$,

$$\int_0^{\alpha}\sin^2\beta\,d\beta = \frac{\alpha}{2}-\frac{\sin\alpha\cos\alpha}{2},\quad \int_{\alpha}^{\pi}\sin^2\beta\,d\beta = \frac{\pi-\alpha}{2}+\frac{\sin\alpha\cos\alpha}{2}.$$

Substituting these values, we find that the average area of the triangle PQR is

$$(\Delta) = \frac{r^2}{2\pi}\int_0^{\pi}\sin^4\alpha\left\{\frac{2}{3}\sin^3\alpha+\cos\alpha\left(\tfrac{1}{2}\pi-\alpha+\sin\alpha\cos\alpha\right)\right\}d\alpha$$

$$= \frac{r^2}{2\pi}\int_0^{\pi}\left(\frac{6}{5}\sin^5\alpha-\frac{1}{3}\sin^7\alpha\right)d\alpha,$$

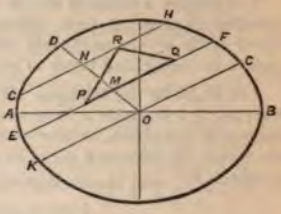
since $\displaystyle\int_0^{\pi}\sin^4\alpha\cos\alpha\,d\alpha = 0$, and $\displaystyle\int_0^{\pi}\sin^4\alpha\cos\alpha\,\alpha d\alpha = -\tfrac{1}{2}\int_0^{\pi}\sin^5\alpha\,d\alpha$.

Hence, finally, the average area of the triangle PQR is

$$(\Delta) = \frac{r^2}{\pi}\int_0^{\frac{1}{2}\pi}\left(\frac{6}{5}\sin^5\alpha-\frac{1}{3}\sin^7\alpha\right) = \frac{r^2}{\pi}\left(\frac{6}{5}\cdot\frac{4.2}{5.3}-\frac{1}{3}\cdot\frac{6.4.2}{7.5.3}\right) = \frac{256r^2}{525\pi}.$$

2. The solution to the problem is the same as the above, when the triangle

PQR is drawn on the surface of any *ellipse*. For suppose, in the accompanying figure, the triangles, of which PQR is one, to have each one side parallel to the diameter COK of the ellipse ADB, OD being the semi-diameter conjugate to OC; then, by precisely the same method as that already employed, we obtain as the average area of all the triangles



$$(\Delta) = \frac{256}{525\pi}\,OC\,.\,OD\sin COD.$$

But the area of the ellipse is

$$\pi\,.\,OC\,.\,OD\sin COD;$$

hence, whatever be the direction of the given line to which COK and PQ are to be parallel, the average area of the triangle PQR, in parts of the area of the ellipse, is $\frac{256}{525\pi^2}$, or ·0494..., that is $\frac{1}{20}$ nearly.

[The value of (Δ) when the three points P, Q, R may, with equal probability, take *any positions* within the ellipse, in parts of the area of the ellipse, has been shown by Mr. Woolhouse and others to be $\frac{35}{48\pi^2}$, or ·07388. See *Reprint*, Vol. VIII., pp. 88 and 105, and Vol. IX., p. 97.]

II. *Solution by* Stephen Watson.

Let CK be a diameter parallel to the given line (See Fig. to preceding Solution), and OD its conjugate; also draw the chords EF, GH, parallel to CK, and cutting OD in M, N. Let a, b be the semi-axes of the ellipse, $OC=a_1$, $OD=b_1$, $OM=y$, $ON=z$, and $\angle DOC=\phi$; then we have

$$EF = \frac{2a_1}{b_1}(b_1^2-y^2)^{\frac{1}{2}}, \qquad GH = \frac{2a_1}{b_1}(b_1^2-z^2)^{\frac{1}{2}}.$$

Now if two points P and Q take every position on EF, the sum of the distances PQ is $\frac{1}{3}EF^3$; hence, EF remaining fixed and R being a point on GH, the sum of the areas of the triangle PQR, while R takes every position in the elliptic segment EDF, is

$$\int_y^{b_1} \tfrac{1}{6}EF^3(z-y)\sin\phi\,.\,GH\,.\,d(z\sin\phi) = \frac{a_1}{3b_1}EF^3\sin^2\phi\int_y^{b_1}(z-y)(b_1^2-z^2)^{\frac{1}{2}}\,dz$$

$$= \frac{a_1}{3b_1}EF^3\sin^2\phi\left\{(\tfrac{1}{3}b_1^2+\tfrac{1}{6}y^2)(b_1^2-y^2)^{\frac{1}{2}}+\tfrac{1}{2}b_1^2y\left(\sin^{-1}\frac{y}{b_1}-\frac{\pi}{2}\right)\right\} \ldots (1).$$

Similarly, when R lies in the segment EBF, the sum of the areas is

$$\frac{a_1}{3b_1}EF^3\sin^2\phi\left\{(\tfrac{1}{3}b_1^2+\tfrac{1}{6}y^2)(b_1^2-y^2)^{\frac{1}{2}}+\tfrac{1}{2}b_1^2y\left(\sin^{-1}\frac{y}{b_1}+\frac{\pi}{2}\right)\right\} \ldots (2).$$

Hence, when R takes every position in the ellipse, and EF every position parallel to CK, the sum of the areas is

$$2\int_0^{b_1}\{(1)+(2)\}\,d(y\sin\phi)$$

$$=\frac{32a_1^4\sin^3\phi}{3b_1^4}\int_0^{b_1}\left\{(\tfrac{1}{2}b_1^2+\tfrac{1}{8}y^2)(b_1^2-y^2)^{\frac{3}{2}}+\tfrac{1}{2}b_1y(b_1^2-y^2)^{\frac{1}{2}}\sin^{-1}\frac{y}{b_1}\right\}dy$$

$$=\frac{4096}{1575}a_1^4b_1^3\sin^3\phi=\frac{4096}{1575}a^2b^2a_1^2b_1\sin\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

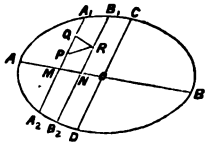
The number of triangles formed is

$$2ab\pi\int_0^{b_1}\mathrm{EF}^2.\,d(y\sin\phi)=\frac{16}{3}ab\,a_1^2b_1\pi\sin\phi;$$

therefore the required average $=\dfrac{256ab}{525\pi}=\dfrac{256}{525\pi^2}$ (area of ellipse),

which is independent of the direction of the given line.

III. *Solution by* SAMUEL BILLS.



In the accompanying diagram, let ABCD be an ellipse, O being the centre, CD a diameter to which one of the sides of each of the triangles is to be parallel, and AB conjugate to CD. Draw A_1A_2, B_1B_2 parallel to CD, and let PQR be one of the triangles. Let A_1A_2, B_1B_2 respectively intersect AO in M and N. Put $AO=a_1$, $CO=b_1$, $OM=a_1\cos\theta_1$, $A_1M=b_1\sin\theta_1$, $ON=a_1\cos\theta_2$, $B_1N=b_1\sin\theta_2$, and angle $AOC=\alpha$. For shortness, put s_1 for the sine and c_1 for the cosine of θ_1; s_2, c_2 for the sine and cosine of θ_2; and s for the sine of α. Let $A_1P=u$ and $A_1Q=v$. Now when Q is *above* P we shall have area of $PQR=\frac{1}{2}a_1s(u-v)(c_1-c_2)$; and when Q is *below* P the area will $=\frac{1}{2}a_1s(v-u)(c_1-c_2)$. Hence the sum of the areas, while Q ranges from A_1 to A_2, will be $=\frac{1}{2}a_1(c_1-c_2)s\left\{\int_0^u(u-v)\,dv+\int_u^{2b_1s_1}(v-u)\,dv\right\}$

$$=a_1(c_1-c_2)s(b_1^2s_1^2-b_1s_1u+\tfrac{1}{2}u^2).$$

And while P also ranges from A_1 to A_2, the sum of the areas will be

$$a_1(c_1-c_2)s\int_0^{2b_1s_1}(b_1^2s_1^2-b_1s_1u+\tfrac{1}{2}u^2)\,du=\tfrac{4}{3}(c_1-c_2)a_1b_1^3ss_1^3;$$

and if R also ranges from B_1 to B_2, the sum of the areas is

$$\tfrac{8}{3}a_1b_1^4ss_1^3s_2(c_1-c_2).$$

Hence, if Σ denote the total sum of the areas for the whole ellipse, we have

$$\Sigma=\frac{16}{3}a_1^3b_1^4s\int_0^{\pi}\int_0^{\theta_2}s_1^3s_2(c_1-c_2)\,d(1-c_2)\,d(1-c_1)$$

$$=\frac{16}{3}a_1^3b_1^4s\int_0^{\pi}\int_0^{\theta_2}s_1^4s_2^2(c_1-c_2)\,d\theta_2\,d\theta_1=\frac{16}{3}\cdot\frac{256}{525}a_1^3b_1^4s \quad \ldots\ldots (1).*$$

Again; to find the number N of triangles. The number, while P and Q each range from A_1 to A_2, and R ranges from B_1 to B_2, will evidently be $8b_1^3 s_1^2 s_2$; hence the total number is

$$N = 16a_1^2 b_1^3 \int_0^{\frac{\pi}{2}} \int_0^{\theta_2} s_1^2 s_2 d(1-c_2)\, d(1-c_1)$$

$$= 16a_1^2 b_1^3 \int_0^{\frac{\pi}{2}} \int_0^{\theta_2} s_1^3 s_2^2 d\theta_2 d\theta_1 = \frac{16}{3}\pi a_1^2 b_1^3 \ldots\ldots\ldots\ldots (2).*$$

Hence the required average is

$$\frac{\Sigma}{N} = \frac{16 \cdot 256 a_1^3 b_1^4 s}{3 \cdot 525} \div \frac{16\pi a_1^2 b_1^3}{3} = \frac{256 a_1 b_1 s}{525\pi} = \frac{256ab}{525\pi},$$

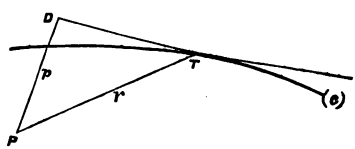
where a, b are the semi-axes of the ellipse.

* (1) and (2) were each doubled because B_1B_2 may be on *either* side of A_1A_2.

2732. (Proposed by W. K. CLIFFORD, B.A.)—An epi- or hypo-cycloid is pushed through a very short fixed tube, so as to remain in one plane, show that the locus of its centre is an ellipse.

Solution by W. S. BURNSIDE, M.A.

In general, if $r = fp$, where r is the radius vector and p the length of the perpendicular on the tangent, be the equation of the given curve, the locus of the pole P is given by the equation $(x^2+y^2)^{\frac{1}{2}} = f(x)$, when the curve (C) always touches the tangent TD at the point T.



In this particular case, $$r^2 = a^2 + \frac{4m}{(m+1)^2} p^2$$

(Salmon's *Higher Plane Curves*, Art. 227); whence the locus of the centre of the hypocycloid, which is the pole now, is given by the equation

$$(m+1)^2(x^2+y^2) - 4mx^2 = (m+1)^2 a^2, \text{ or } \frac{x^2}{(m+1)^2} + \frac{y^2}{(m-1)^2} = b^2,$$

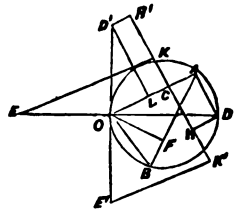
as $a+b = mb$, a being the radius of the fixed circle, and b the radius of the moving circle, in the description of the hypocycloid.

2100. (Proposed by the EDITOR.)—1. From one end O of a fixed diameter OD of a circle, two chords OA and OB are drawn such that the

rectangle OA . OB = a constant (k^2); show that the envelope of the line AB is a circle with centre O and radius a third proportional to OD and k.

2. On a chord OA of a circle OAD, a point C is taken such that the difference or (3) the sum of the squares on OC and OA = a constant (k^2); show that the envelope of the perpendicular drawn to OC through C is a conic having O as centre. [From the *Nouvelles Annales de Mathématiques.*]

I. *Solution by* MORGAN JENKINS, B.A.



1. Draw OF perpendicular to AB, produced if necessary; then the triangles OFB, OAD are similar, and OF : OB = OA : OD; hence OF is constant, and therefore the envelope is a circle with centre O [and radius a third proportional to OD and k, since OF . OD = k^2.]

2. Produce DO to E, making OE = OD; draw DH, EK parallel to OA, and through C draw KCH perpendicular to OA and meeting DH, EK in H, K respectively. Then DH . EK = the difference of the squares on OC and OA = k^2; therefore the envelope of KCH is in this case a central conic with D and E as foci, the conic being an ellipse or hyperbola according as OC is greater or less than OA.

3. When the *sum* of the squares on OA and OC = the constant k^2, draw D'OE' a tangent to the circle making OD' = OE' = OD, and draw D'L, D'H', perpendicular to OA, HK.

Then from similar triangles D'OL, ODA, we have OL = AD, and

$$D'H' \cdot E'K' = OC^2 \sim OL^2 = OC^2 \sim AD^2 = OA^2 + OC^2 \sim OD^2 = k^2 \sim OD^2.$$

And it will be seen that the perpendiculars D'H', E'K' are on the same or opposite sides of H'K' according as k > or < OD. Hence the envelope of H'K' is an ellipse or hyperbola with D', E' for foci according as k is > or < OD. When k = OD, the envelope reduces to the two points D', E'; and this is at once seen to be true by observing that the perpendicular from either of these points on OC forms with OC and D'OE' a triangle equal in all respects to the triangle OAD.

II. *Solution by* J. J. WALKER, M.A.

1. The envelope of the secant AB is obviously a circle with O as centre, since in any triangle the length of the perpendicular on the base is the fourth proportional to the diameter of the circumscribing circle and the two sides [in this case a third proportional to OD and k].

2. Let OD = d, and $\angle AOD = \phi$; and let (ρ, θ) be the polar coordinates of a point in HCK, referred to O as pole and OD as polar axis. Then we have

$$\rho^2 \cos^2(\phi - \theta) = OC^2 = OA^2 \mp k^2 = d^2 \cos^2\phi \mp k^2,$$

or

$$(\rho^2 \sin^2\theta \pm k^2) \tan^2\phi + \rho^2 \sin 2\theta \tan\phi + \rho^2 \cos^2\theta - d^2 \pm k^2 = 0 \ldots\ldots (1).$$

Differentiating with respect to the parameter $\tan\phi$, we have

$$2(\rho^2 \sin^2\theta \pm k^2) \tan\phi + \rho^2 \sin 2\theta = 0 \ldots\ldots\ldots\ldots (2).$$

Finally, eliminating $\tan\phi$ between (1) and (2), we have

$$\rho^2 \left\{ k^2 \cos^2\theta \mp (d^2 \mp k^2) \sin^2\theta \right\} = (d^2 \mp k^2) k^2 \ldots\ldots\ldots\ldots (3)$$

as the equation to the required envelope (the upper or lower sign being used according as OC is taken less or greater than OA); and it is plainly that of a central conic, having O as its centre and D as one of its foci; and being a hyperbola if OC be taken less than OA, an ellipse if otherwise.

[A slight variation of Mr. WALKER'S method may be obtained by writing the equation of HCK in the form (taking the case when the *sum* of the squares on OA and OC$=k^2$) $(x\cos\phi+y\sin\phi)^2+d^2\cos^2\phi=k^2$,

and, putting $\tan\phi=t$, this equation becomes

$$(x+yt)^2-k^2t^2\equiv\left\{x+(y+k)t\right\}\left\{x+(y-k)t\right\}=k^2-d^2\ldots\ldots(\alpha).$$

Differentiating (α) with respect to t, we have

$$t=\frac{xy}{k^2-y^2},\quad x+(y+k)t=\frac{kx}{k-y},\quad x+(y-k)t=\frac{kx}{k+y};$$

hence, substituting in (α), the equation of the envelope is found to be

$$\frac{x^2}{k^2-d^2}+\frac{y^2}{k^2}=1\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta),$$

a conic (ellipse or hyperbola according as $k>$ or $<d$) having O as centre, and D′, E′ as foci.]

2756. (Proposed by J. GRIFFITHS, M.A.)—Show that an infinite number of triangles can be described such that each has the same circumscribing, nine-point, and self-conjugate circles as a given triangle.

Solution by Professor CAYLEY.

It is a known theorem that if two triads of points, say A, B, C and A′, B′, C′, are self-conjugate in regard to a conic S, they lie in a conic Σ. Take the conic S and the points A, B, C as given; then Σ will be a conic passing through A, B, C; and if on this conic we take any point A′, and then take B′ to be either of the intersections of the conic Σ by the polar of A in regard to S, and finally construct C′ as the pole of A′B′ in regard to S, then, by what precedes, C′ will be on a conic through A, B, C, A′, B′, that is, on the conic Σ. Or given the conic S, the triangle ABC, and the conic Σ through A, B, C, we obtain an infinity of triangles A′B′C′, self-conjugate in regard to S and inscribed in Σ. That is, if S, Σ are circles, we have an infinity of triangles self-conjugate in regard to the circle S and inscribed in the circle Σ; and inasmuch as the nine-points circle can be constructed by means of the two circles S, Σ alone, the triangles have all of them the same nine-points circle.

2334. (Proposed by W. S. BURNSIDE, M.A.)—1. Construct a triangle of given species, having its vertices on three concentric circles.

2. The triangle being equilateral, what relation exists between the radii when only one such triangle can be so constructed?

I. *Solution by* W. S. McCay, B.A.

1. The problem is the same as to find a point the ratios of whose distances from the vertices of a triangle are given; for then the figure may be enlarged to scale. Now when we have given the base and ratio of the sides of a triangle, the locus of the vertex is a circle drawn on the segment of the base that cuts it harmonically in the given ratio; hence the required point must lie on three such circles. But since the ratios of section of the three sides are connected (from the nature of the question) by the relation $l \cdot m \cdot n = 1$, therefore three of the points of section connect through a point to the opposite vertices, and the other three lie on a right line; hence the three segments may be considered as the diagonals in magnitude and position of a complete quadrilateral; and we know that the three circles described on these have a common radical axis; so there are, in general, *two* solutions to the question.

2. If there were only one solution, the point must lie on the circumscribing circle of the triangle; for it is well known that this circle cuts the other three orthogonally; therefore, if the three circles touch each other, the circumscribing circle must go through the point of contact. Hence, if α, β, γ be the radii of the given circles, and $a : b : c$ be the given ratio of sides of the triangle, Ptolemy's theorem gives the relation $a\alpha \pm b\beta \pm c\gamma = 0$.

II. *Solution by the* Proposer.

1. Let the radii of the three circles be denoted by α, β, γ, and the sides of the required triangles by $\lambda a, \lambda b, \lambda c$; then, making use of the relation which exists between the six connectors of four points in a plane, we have

$$\lambda^4 a^2b^2c^2 + \lambda^2 \begin{vmatrix} a^2\alpha^2 & a^2(a^2-b^2-c^2) \\ b^2\beta^2 & b^2(b^2-c^2-a^2) \\ c^2\gamma^2 & c^2(c^2-a^2-b^2) \end{vmatrix} + \begin{vmatrix} a^2(a^2-\beta^2) & a^2(a^2-\gamma^2) \\ b^2(\beta^2-a^2) & b^2(\beta^2-\gamma^2) \\ c^2(\gamma^2-a^2) & c^2(\gamma^2-\beta^2) \end{vmatrix} = 0,$$

an equation which in general gives two distinct values for λ, and therefore two solutions of the question.

2. The discriminant of the above equation is found to be

$$M(b\beta + c\gamma - a\alpha)(c\gamma + a\alpha - b\beta)(a\alpha + b\beta - c\gamma),$$

where $$M = (b+c-a)(c+a-b)(a+b-c),$$

which vanishes when $a\alpha \pm b\beta \pm c\gamma = 0$, the geometric interpretation of which is obvious. Hence, if $a = b = c$, we have $\alpha \pm \beta \pm \gamma = 0$, the required relation between the radii.

2474. (Proposed by Professor Everett.)—

1. If $\left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 = 0$, show that, when n is any positive integer,

$$\left(\cos\theta \cdot \frac{d}{dx} + \sin\theta \cdot \frac{d}{dy}\right)^n = \left(\cos n\theta \cdot \frac{d}{dx} + \sin n\theta \cdot \frac{d}{dy}\right)\left(\frac{d}{dx}\right)^{n-1};$$

and hence (2) prove the theorem (due to Rankine and Stokes) that at every multiple point in a plane curve which fulfils the condition $\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0$, the branches make equal angles with each other.

Solution by the PROPOSER.

1. If $a^2+b^2=0$, we have $b=ia$ [where $i=\sqrt{(-1)}$], and

$$(a\cos\theta+b\sin\theta)^n = a^n(\cos\theta+i\sin\theta)^n = a^n(\cos n\theta+i\sin n\theta)$$
$$= a^{n-1}(a\cos n\theta+b\sin n\theta).$$

Hence, writing $\frac{d}{dx}$ for a and $\frac{d}{dy}$ for b, the required equation is proved.

It may also be proved directly without the use of imaginaries, as follows:

$\left(\cos A\,.\,\frac{d}{dx}+\sin A\,.\,\frac{d}{dy}\right)\left(\cos B\,.\,\frac{d}{dx}+\sin B\,.\,\frac{d}{dy}\right)$ is, by actual development,

$$=\left\{\cos(A+B)\frac{d}{dx}+\sin(A+B)\frac{d}{dy}\right\}\frac{d}{dx}, \quad \text{since } \left(\frac{d}{dy}\right)^2=-\left(\frac{d}{dx}\right)^2.$$

If $A=B=\theta$, this becomes

$$\left(\cos\theta\,.\,\frac{d}{dx}+\sin\theta\,.\,\frac{d}{dy}\right)^2=\left(\cos 2\theta\,.\,\frac{d}{dx}+\sin 2\theta\,.\,\frac{d}{dy}\right)\frac{d}{dx}.$$

Hence the equation holds when $n=2$, and by putting $A=(m-1)\theta$, $B=\theta$, we may show that, if it holds for any one integral value of n, it holds for the next greater.

2. To apply this result to the proof of Rankine and Stokes' theorem. At a point where n branches of any curve $\phi(x,y)=0$ cut each other, the condition $\left(dx\frac{d}{dx}+dy\frac{d}{dy}\right)^n\phi=0$ is fulfilled, which may also be written $\left(\cos\theta\,.\,\frac{d}{dx}+\sin\theta\,.\,\frac{d}{dy}\right)^n\phi=0$, θ denoting the inclination of any branch to the axis of x. When $\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}=0$, this condition reduces to

$$\left(\cos n\theta\,.\,\frac{d}{dx}+\sin n\theta\,.\,\frac{d}{dy}\right)\left(\frac{d}{dx}\right)^{n-1}\phi=0,$$

and all the values of θ that satisfy it must give the same value to the ratio $\sin n\theta : \cos n\theta$, that is to say, $\tan n\theta$. Hence the values of $n\theta$ have a common difference π, and the values of θ a common difference, that is to say, the angle between any two consecutive branches is $\frac{\pi}{n}$.

2503. (Proposed by H. R. GREER, B.A.)—Prove that the number of normals that can be drawn from a point in space of p dimensions to a (hyper)$^{p-3}$ surface (i. e., what corresponds to a *surface* in space of three dimensions) of the nth degree is $\frac{n}{n-2}\left\{(n-1)^p-1\right\}$.

Solution by the REV. J. WOLSTENHOLME, M.A.

In the same way as the number of normals is found in ordinary three

dimensions, we have, if U_p be the number drawn from a fixed point to a surface of nth degree, in hyper-geometry of p dimensions,

$$U_p = n(n-1)^{p-1} + U_{p-1},$$

the first term of the right-hand number being the number of normals parallel to one of the axes of coordinates which meet the surface at a finite distance; and the second the number which meet it at infinity. Hence we have

$$U_p - U_{p-1} = n(n-1)^{p-1}, \quad U_{p-1} - U_{p-2} = n(n-1)^{p-2}, \quad \ldots\ U_2 = n^2;$$

therefore $U_p = n - 1 + (n-1) + (n-1)^2 + \ldots + (n-1)^{p-1} = \dfrac{n\{(n-1)^p - 1\}}{n-2}$.

[Mr. GREER sends the following *Note* on this Question:—

"Take the case of three dimensions analytically; then, taking origin as the point whence normals are drawn, and writing $\dfrac{dU}{dx} = A$, $\dfrac{dU}{dy} = B$, $\dfrac{dU}{dz} = C$,

we have $U = 0$ (equation of surface of nth degree), $\dfrac{x}{A} = \dfrac{y}{B} = \dfrac{z}{C}$ ....(1),

or $U = 0, \quad xB - yA = 0, \quad xC - zA = 0$ .....(2),

and, at first sight, we should *expect* the number of solutions to be n^3. The question is, how the reduction arises?"

In reference to this point, Mr. WARREN remarks as follows:—

"Although *all* the solutions of system (1) are contained in system (2), *the reverse is not true*; for system (2) is satisfied by ($U=0$, $x=0$, $A=0$). As ($yC - zB$) is not however *in general* rendered zero by ($x=0$, $A=0$, $U=0$), we must subtract the number of solutions of ($U=0$, $A=0$, $x=0$), that is $n(n-1)$, from the number of solutions of ($U=0$, $xB - yA = 0$, $xC - zA = 0$), that is n^3, in order to get the number of values that satisfy the system (1), which is therefore $n^3 - n^2 + n$. For $n=2$, we get the number $= 6$, which is true. The problem is a particular case of a very interesting class investigated first by CAYLEY, and extended by SALMON and S. ROBERTS."

See Frost and Wolstenholme's *Solid Geometry*, Art. 341, and Hymers' *Geometry of Three Dimensions*, p. 238, ex. 12.]

2682. (Proposed by A. W. PANTON, A.B.)—If circles be described on a pair of opposite sides of a square; prove geometrically that

1. The polars of any point on either diagonal with respect to the two circles meet on the other diagonal.
2. The four tangents from the point form a harmonic pencil.

Solution by PRINCE DE POLIGNAC.

The two parts of the question can be demonstrated separately without much difficulty. In what follows I propose to set forth their connection, and to show that both must necessarily coexist and depend upon the same general principle.

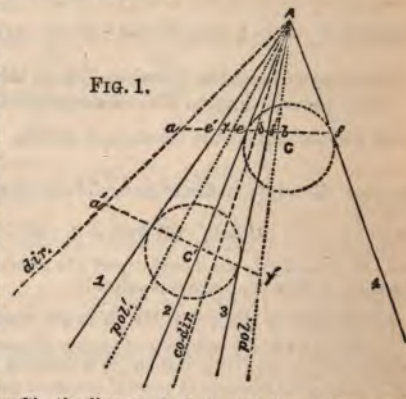
When four tangents to two conics form a harmonic pencil, the tangents to the same conics being conjugate, it is clear that, taking any pair of tangents to either conic, the pole of any one of the tangents with respect to the *other* conic lies on the *other* tangent.

This is merely another way of stating the same fact, and can be looked upon as a graphic criterion by means of which the harmonic property of the pencil can be tested.

Now it is easy to see that this criterion is only a particular case of a more general rule, which may be stated as follows; viz.,

Four lines, 1, 2, 3, 4, being given (Fig. 1), issuing from a point A; draw any two circles (or conics) C and C′ touching 2, 4 and 1, 3, respectively. Draw also through A an arbitrary line which we shall call directrix, and take its intersections a and a' with the polars of A with respect to C and to C′. Next draw the polars of a with respect to C, and of a' with respect to C′, and take their respective intersections with the polars of A with regard to C′ and to C; call them γ' and γ.

FIG. 1.



Now, if the polar of γ with respect to C coincides with the polar of γ' with respect to C′; the lines 1, 2, 3, 4 will form a harmonic pencil, and otherwise not.

Demonstration.—Let us call Co-directrix the common polar of γ and γ'. If we write *pol.* and *pol.* for the polar of a with respect to C, and that of a' with respect to C′, we shall have 8 lines issuing from A, viz.,

dir., 1, pol.′, 2, co-dir., 3, pol., 4.

Using, for shortness' sake, the notation $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ to express the harmonic ratio where the conjugate points or lines are written *under* each other, we have obviously the following harmonic pencils and corresponding points in harmonic ratio on any transversal, say the polar of A with respect to C.

$$(C)\quad \left\{ \begin{matrix} \begin{bmatrix} 2 \text{ dir.} \\ 4 \text{ pol.} \end{bmatrix} & \begin{bmatrix} e & a \\ f & b \end{bmatrix}; & \begin{bmatrix} 1 \text{ dir.} \\ 3 \text{ pol.}' \end{bmatrix} & \begin{bmatrix} e' & a \\ f' & \gamma \end{bmatrix} \\ \begin{bmatrix} 2 \text{ pol.}' \\ 4 \text{ co-dir.} \end{bmatrix} & \begin{bmatrix} e & \gamma \\ f & \delta \end{bmatrix}; & \begin{bmatrix} 1 \text{ pol.} \\ 3 \text{ co-dir.} \end{bmatrix} & \begin{bmatrix} e' & b \\ f' & \delta \end{bmatrix} \end{matrix} \right\} (C').$$

Likewise, if we use the notation $\begin{Bmatrix} a & c & e \\ b & d & f \end{Bmatrix}$ to express involution, the conjugate points being written *under* each other; the harmonic system (C) will give, through well-known properties of involution,

$$\begin{Bmatrix} e & a & \gamma \\ e & b & \delta \end{Bmatrix} \text{ or } \begin{Bmatrix} f & a & \gamma \\ f & b & \delta \end{Bmatrix}.$$

The system (C′) gives $\begin{Bmatrix} e' & a & \gamma \\ f' & b & \delta \end{Bmatrix}$;

hence, as is well known, $\begin{matrix} e \\ f \end{matrix}$ foci of $\begin{Bmatrix} e' & a & \gamma \\ f' & b & \delta \end{Bmatrix}$.

Therefore $\begin{bmatrix} e & e' \\ f & f' \end{bmatrix}$ or $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$;

that is to say, the four lines 1, 2, 3, 4 form a harmonic pencil.

Conversely, if 1, 2, 3, 4 form a harmonic pencil, we shall have the following four harmonic ratios:

$$\begin{bmatrix} e & a \\ f & b \end{bmatrix} \quad \begin{bmatrix} e' & a \\ f' & \gamma \end{bmatrix} \quad \begin{bmatrix} e & \gamma \\ f & \delta \end{bmatrix} \quad \begin{bmatrix} e' & e \\ f' & f \end{bmatrix}$$

From the second and first systems we draw the inferences

$$\begin{Bmatrix} e' & a & e \\ e' & \gamma & f \end{Bmatrix} \text{ and } \begin{Bmatrix} f' & a & e \\ f' & \gamma & f \end{Bmatrix}; \quad \begin{Bmatrix} e & a & b \\ f & \gamma & \delta \end{Bmatrix}.$$

Hence $\begin{matrix} e' \\ f' \end{matrix}$ foci of $\begin{Bmatrix} e & a & b \\ f & \gamma & \delta \end{Bmatrix}$; therefore $\begin{bmatrix} e' & b \\ f' & \delta \end{bmatrix}$,

which, as in (C′), expresses that the polar of γ' with regard to circle C′ coincides with that of γ with regard to C; in other words, that a co-directrix is obtained.

If we take the co-directrix as directrix, these two lines will interchange. So will pol. and pol.′; and we can state the rule thus:—

Draw through A any directrix. Take its pole γ with regard to C. Take the polar of γ with respect to C′, and its intersection a' with the polar of A with respect to C′. Draw Aa' the co-directrix. Now resume the construction *mutatis mutandis*, *i.e.*, beginning with the same directrix and interchanging the accents, or beginning with the co-directrix without changing the accents. The same final lines will be obtained if the four tangents form a harmonic pencil, and conversely.

If one of the tangents, say 2, be taken as directrix, the harmonic symbols (C) and (C′) show that pol. must also coincide with 2, and that both pol.′ and co-dir. coincide with the conjugate tangent 4. This particular case gives the remark made at the beginning.

When the lines 1, 2, 3, 4 form a harmonic pencil, if we call A′ the point of intersection of the two polars of A, and take AA′ as directrix, we shall obtain two points γ, γ' which will determine the co-directrix. Now, if we resume the construction from A′ with the same directrix A′A, we shall obtain all the same lines in an inverted order, and the same two points γ, γ'; hence, if the polar of γ with regard to C′ coincides with the polar of γ' with regard to C, the four tangents from A′ will also form a harmonic pencil; and conversely.

In the case of the given question (Fig. 2), the middle points of the horizontal sides of the square have inversely the same two polars with regard to both circles. These polars are the two diagonals. This is sufficient to show that if a point A be taken on any diagonal, and if (1) in the given question is true for that point, (2) is true also, and *vice versâ*. Because, if we take the diagonal as directrix, and apply the last statement of the rule, the middle points of horizontal sides will be our points γ and γ', and we have to take the intersections of the polars

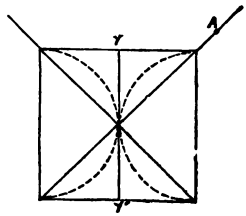


FIG. 2.

of γ and γ', *i.e.* the *second diagonal*, with the polars of A respectively. But if (1) is true for A, both intersections coincide; that is to say, we obtain a co-directrix. Hence (2) is true, and the converse holds.

Furthermore, if we take (Fig. 3) any number of points A, A_1, &c., on one diagonal, and call p, p_1, &c., p', p'_1, &c., the respective intersections of their polars with the other diagonal; if the points p, p_1 &c., do not coincide with p', p'_1, &c., they will be homographic, as to one point A corresponds only *one* point p and *one* point p', and *vice versâ*. If O, centre of the square, be taken as origin, it is obvious that when $Op = \infty$, $Op' = \infty$, and when $Op = 0$, $Op' = 0$, hence $\frac{Op}{Op'}$ = constant. Consequently, either the two conjugate points will always, or they will never, coincide.

FIG. 3.

In other words,—

If (1) *or* (2) *be true for one position of* A, (1) *and* (2) *will be true for every position of* A.

Now the truth of the given question can be ascertained in various ways. For instance, A may be taken at one vertex of the square. Then its polars evidently meet at the other vertex on the common tangent. Hence, &c.

Or A may be taken at infinity, and (2) directly verified from the intercepts made by tangents 1, 3; 1, 2, and 1, 4, on the diagonal perpendicular to their direction. Taking the radius as unity, these are at once found to be 2, $\sqrt{2}$, $2+\sqrt{2}$, and obviously $\frac{2}{\text{intercept}\,(1,3)} = \frac{1}{(1,2)} + \frac{1}{(1,4)}$.

A converse theorem can also be proved; viz., that if the polars of A with respect to two circles meet in A′, and if the four tangents from A as well as from A′ form harmonic pencils (tangents to the same circles being conjugate), the circles must be equal and must also touch each other.

It follows from what has been said, that in this case there must be in the plane two such points as γ, γ' whose polars with respect to the two circles are inversely coincident. Now it is clear that the figure $C\gamma C'\gamma'$ must be a parallelogram (Fig. 3), and if we call p, q the distances from C to the two polars, and the same accented from C′, and put

$$CC' = c, \quad C\gamma = l, \quad C\gamma' = l', \quad C'\hat{C}\gamma = \gamma, \quad C'\hat{C}\gamma' = \gamma',$$

we have at once $pl = r^2$, $ql' = r^2$, $p'l = r'^2$, $q'l' = r'^2$ ..... (1, 2, 3, 4);

hence $\frac{p}{p'} = \frac{q}{q'}$. This result shows that the points of intersection of $\overline{CC'}$ with the two polars either coincide or divide CC′ harmonically. But it is easy to see that they must coincide, from the consideration that the centre can never lie between the pole and its polar. Therefore, calling g the distance from C to the intersection of the two polars, we have the additional equations

$$\frac{p}{p+p'} = \frac{g}{c}, \quad p = g\cos\gamma, \quad q = g\cos\gamma' \ldots\ldots\ldots (5, 6, 7),$$

$$l\cos\gamma + l'\cos\gamma' = c, \quad l\sin\gamma = l'\sin\gamma' \ldots\ldots\ldots\ldots (8, 9).$$

(1) and (2) give $pl = ql'$, which, combined with (6), (7), and (9), leads to

$$\gamma = \gamma', \quad l = l', \quad p = q, \quad c = 2l\cos\gamma;$$

and the former equations reduce to

$$pl = r^2, \quad p'l = r'^2, \quad p + p' = c\cos\gamma, \quad c = 2l\cos\gamma \ldots\ldots (1\ldots4).$$

(1), (2), and (3) give at once $r^2 + r'^2 = cl\cos\gamma$, which, combined with (4), gives $c^2 = 2(r^2 + r'^2) = (r+r')^2 + (r-r')^2$, therefore $c > r + r'$.

If r and r' be given and c determined from the above equation, the circles cannot meet in real points, and a locus will be obtained for γ and γ', viz., the perpendicular drawn to the line joining centres half-way between them. γ and γ' will be equidistant from the line joining centres, and it is clear that the point at which the polars of γ and γ' meet has the line $\gamma\gamma'$ for polar with respect to both circles.

Now two points γ, γ' being selected, if two points A, A′ can be found whence the tangents form harmonic pencils, they must lie on the polars of γ and γ', each of which here becomes the directrix (or co-directrix). And it will be seen as above that, taking any number of points on the 1st directrix, the intersections of their polars with the 2nd directrix will be homographic, so that A′ is the point which is its own conjugate, and the same for A. But these points are in both systems, the point (g) where the two directrices meet, and the points A and A′ where these lines meet γ, γ'. If γ and γ' are taken outside the circle, A and A′ lie inside; if γ, γ' are taken inside the circle, the tangents which should be conjugates lie outside of the other two; therefore no harmonic pencils can be obtained, except in the limiting case where $r = r'$, as then $c = r + r'$; that is to say, the circles are equal and touch each other.

It will also be seen that, in the general case, although A and A′ are *real* points, the harmonic symbols in the systems (C) and (C′) reduce to two, therefore no inference can be drawn.

2748. (Proposed by W. K. Clifford, B.A.)—If a circular cubic with a double point O be cut by a circle in four points A, B, C, D; and if OA, OB, OC, OD cut the circle again in E, F, G, H; show that any pair of straight lines joining these four points will be equally inclined to the bisectors of the angles between the tangents at O.

I. *Solution by* Archer Stanley.

This theorem is the inverse of the following well known one:—

Any two opposite connectors of the four points common to a circle and a conic are equally inclined to the axes of the latter.

For if the double point O of the circular cubic be taken as the centre of inversion, the inverse curve will be a conic passing through O, whose asymptotes are parallel to the tangents at the double point, and whose axes, therefore, are parallel to the bisectors of the angles made by these tangents. Again, the inverse of the circle cutting the cubic in A, B, C, D will be another circle cutting the conic in A′, B′, C′, D′, and the points E′, F′, G′, H′, respectively inverse to E, F, G, H, will of course be the second intersections of A′O, B′O, C′O, D′O with the circle A′B′C′D′.

Now, by a well known property of two inverse circles, the connector of any two of the four points E, F, G, H is parallel to that of the corresponding

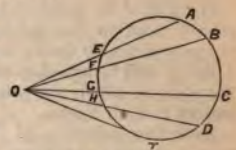
two of the four points A′, B′, C′, D′. (For instance, EF and A B , being each anti-parallel to AB, are parallel to one another). The truth of the proposed theorem is consequently obvious.

II. *Solution by* W. S. McCay, B.A.

The inverse of the cubic with respect to the double point is a conic, whose asymptotes are parallel to the tangents at the double point. The first part is evident from the form of the equation

$$(x^2+y^2)\,u_1+ax^2+2hxy+by^2=0,$$

the origin being at the double point; and the second part follows.



Clearly E, F, G, H are points on the inverse curve if $\rho\rho'=\mathrm{OT}^2$; therefore E, F, G, H are the intersections of the circle with a conic whose axes are parallel to the bisectors of angles between tangents at the double point O.

It appears in the same way that the theorem is equally true for a bicircular quartic; for the curve $(x^2+y^2)^2+(x^2+y^2)\,u_1+ax^2+2bxy+by^2=0$ also inverts into a conic, the origin not being a point on the *conic*, as it is in the other case.

2718. (Proposed by Professor Cayley.)—Find *in plano* the locus of a point P, such that from it two given points A, B, and two given points, C, D subtend equal angles.

I. *Solution by* W. S. Burnside, M.A.

1. Let the length of AB (Fig. 1) be $2a$, and its equation $\alpha=0$; the length of the bisector PM of the base ρ, and the angle $\mathrm{APB}=\theta$. Then we have

$$\mathrm{AB}^2=\mathrm{AP}^2+\mathrm{BP}^2-2\,\mathrm{AP}\,.\,\mathrm{BP}\cos\theta,$$

or $\mathrm{AP}\,.\,\mathrm{BP}\cos\theta=\rho^2-a^2$, also $\mathrm{AP}\,.\,\mathrm{BP}\sin\theta=2a\alpha$;

hence $$\tan\theta=\frac{2a\alpha}{\rho^2-a^2}.$$

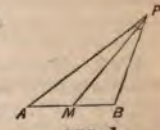


FIG 1.

Replacing ρ^2-a^2 by U, where $\mathrm{U}=0$ is the equation of the circle on AB as diameter, $\tan\theta=\dfrac{2a\alpha}{\mathrm{U}}$. Similarly, if CD subtends an angle η, $\tan\eta=\dfrac{2b\beta}{\mathrm{V}}$.

Hence, if $\theta=\eta$, the equation of the locus is $a\alpha\mathrm{V}=b\beta\mathrm{U}$, which represents a circular cubic. (See Salmon's *Higher Plane Curves*, Art. 141.)

2. In a similar manner, we may investigate the locus of P, when the points P, A, B, C, D are situated on the surface of a sphere (Fig. 2).

In this case we have

$$\tan\theta=\frac{\sin 2a\,.\,\alpha}{\sin^2\rho\,(1-\sin^2 a\sin^2\alpha)-\sin^2 a},$$

where ω is the angle which the bisector ρ of AB makes with AB, and $\mathrm{AB}=2a$ as before. Hence we may write $\tan\theta=\dfrac{\sin 2a\,.\,\alpha}{\mathrm{U}}$, where $\mathrm{U}=0$ is

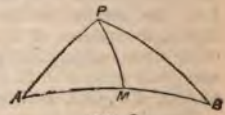


FIG. 2.

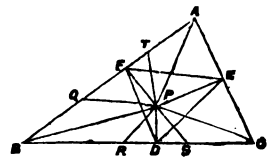
the equation of a sphero-conic. And we have for the equation of the locus

$$\sin 2\alpha \,.\, \alpha V = \sin 2\beta \,.\, \beta U$$

(where α is the sine of the arc through P perpendicular to AB, or $\alpha = 0$ the equation of the plane through AB and the centre of the sphere).

II. *Solution by the* EDITOR.

1. Let BC ($=a$) and BA ($=c$) be two given lines meeting at a given angle at B, and let P be a variable point at which BC and BA subtend equal or supplementary angles; then, drawing PD, PE, PF perpendicular to BC, CA, AB, we have



$$\frac{PD}{PF} = \frac{\sin PBD}{\sin PBF} = \frac{\sin PBC}{\sin BPC} \cdot \frac{\sin BPA}{\sin PBA}$$

$$= \frac{PC}{BC} \cdot \frac{BA}{PA} = \frac{PC \,.\, \sin BCA}{PA \,.\, \sin BAC} = \frac{DE}{EF},$$

therefore

$$PD^2 \,.\, EF^2 = PF^2 \,.\, DE^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now, taking ABC as triangle of reference, let the trilinear coordinates of P be (x, y, z); then the relation (1) gives at once for the equation of the locus of P

$$x^2 (y^2 + z^2 + 2yz \cos A) = z^2 (x^2 + y^2 + 2xy \cos C) \ldots\ldots\ldots (2).$$

Reducing (2), we find that part of the locus is

$$y = 0, \text{ that is to say, the line AC} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

as indeed might have been seen at once without calculation, since BA and BC obviously subtend the same angle at a point in AC produced, and supplementary angles at any point in AC between A and C.

The other part of the locus is the cubic

$$(x^2 - z^2)\, y + 2xz\,(x \cos A - z \cos C) = 0 \quad \ldots\ldots\ldots\ldots (4),$$

which may be written in the form $xV - zU = 0$,

where $U \equiv yz + x(-x \cos A + y \cos B + z \cos C)$,

and $V \equiv xy + z(\ x \cos A + y \cos B - z \cos C)$;

so that $U = 0$, $V = 0$ are the equations of the circles on BC, BA as diameters.

If the angles A and C are equal, or BA = BC, the cubic locus (4) breaks up into the line $x - z = 0$, that is, the bisector of the angle B, and

$$\frac{1}{x} + \frac{2 \cos A}{y} + \frac{1}{z} = 0, \quad \text{or} \quad \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0,$$

which is the equation of the circle circumscribing the triangle ABC.

2. Next suppose CR ($=h$), and AQ ($=k$) to be the fixed lines which subtend equal angles at P; then, since

$$\cot RPC = \cot (RPD + DPC) = \frac{PD^2 - RD \,.\, DC}{PD \,.\, RC},$$

we have, when the angle RPC = QPA, the relation

$$AQ \,.\, PF\,(PD^2 - RD \,.\, DC) = CR \,.\, PD\,(PF^2 - AF \,.\, FQ) \quad \ldots\ldots\ldots (5)$$

The bracketed expressions in (5) are those (U, V, say) which when equated to zero give the equations of the circles on CR, AQ as diameters; hence, for supplementary or equal values of the angles CPR, ABQ, the locus of P is the circular cubic $kx\mathrm{V} \pm kz\mathrm{U} = 0$ ..... (6).

We may express U in various ways: thus, if S, T be the middle points of CR, AQ, we have $\mathrm{RD}.\mathrm{DC} = \mathrm{SC}^2 - \mathrm{SD}^2$, and

$$\mathrm{U} \equiv \mathrm{PD}^2 - \mathrm{RD}.\mathrm{DC} \equiv \mathrm{PC}^2 - \mathrm{RC}.\mathrm{CD} \equiv \mathrm{SP}^2 - \mathrm{SC}^2 \ldots\ldots (7).$$

If we take, as before, ABC as triangle of reference, and put (x, y, z) for the trilinear coordinates of P, we have

$$\mathrm{CD} = x\cot \mathrm{C} + y \operatorname{cosec} \mathrm{C}, \text{ and } \mathrm{CP}^2 . \sin^2 \mathrm{C} = \mathrm{DE}^2 = x^2 + y^2 + 2xy\cos \mathrm{C},$$

therefore $\mathrm{U} \equiv (x^2 + y^2 + 2xy\cos \mathrm{C})\operatorname{cosec}^2 \mathrm{C} - k(x\cot \mathrm{C} + y \operatorname{cosec} \mathrm{C}) \ldots . (8).$

From this we may at once write down the trilinear equation of the locus, and render it homogeneous, if necessary. Under another form the equation of the locus is given in the Solution of Question 1721, on p. 41 of Vol. IV. of the *Reprint*.

3. If the two given lines are in the same straight line, BR and CS in the above figure suppose, the locus of P is a circle having as the ends of a diameter the two points which divide BC internally and externally in the ratio of BM : CN, BM and CN being tangents from B and C to a circle on RS as diameter. This may be very simply shown as follows:—

$$\frac{\mathrm{BP}.\mathrm{PS}}{\mathrm{CP}.\mathrm{PR}} = \frac{\triangle \mathrm{BPS}}{\triangle \mathrm{CPR}} = \frac{\mathrm{BS}}{\mathrm{CR}}, \text{ and } \frac{\mathrm{BP}.\mathrm{PR}}{\mathrm{CP}.\mathrm{PS}} = \frac{\triangle \mathrm{BPR}}{\triangle \mathrm{CPS}} = \frac{\mathrm{BR}}{\mathrm{CS}},$$

therefore

$$\frac{\mathrm{BP}^2}{\mathrm{CP}^2} = \frac{\mathrm{BR}.\mathrm{BS}}{\mathrm{CR}.\mathrm{CS}} = \frac{\mathrm{BM}^2}{\mathrm{CN}^2};$$

hence we have

$$\mathrm{BP} : \mathrm{CP} = \mathrm{BM} : \mathrm{CN},$$

and therefore the locus of P is the above-mentioned circle.

2737. (Proposed by Professor CAYLEY.)—Find *in solido* the locus of a point P, such that from it two given points, A, C, and two given points B, D, subtend equal angles.

Solution by J. J. WALKER, M.A.

By steps similar to those adopted in the Solution of the preceding Question (2718), the equation of the locus is at once found to be

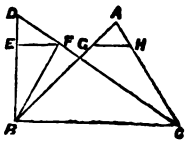
$$a^2\left[\left\{(x-x')^2 + (y-y')^2 + (z-z')^2\right\} - \left\{l(x-x') + m(y-y') + n(z-z')\right\}^2\right] \times \left[(x-x'')^2 + (y-y'')^2 + (z-z'')^2 - b^2\right]^2 =$$

$$b^2\left[\left\{(x-x'')^2 + (y-y'')^2 + (z-z'')^2\right\} - \left\{l'(x-x'') + m'(y-y'') + n'(z-z'')\right\}^2\right] \times \left[(x-x')^2 + (y-y')^2 + (z-z')^2 - a^2\right]^2,$$

where (x', y', z') are the coordinates of the middle point of AB, and (l, m, n) its direction cosines, (x'', y'', z'') and (l', m', n') referring similarly to CD, and (x, y, z) being the coordinates of the point whose locus is sought—a surface of the sixth degree.

2485. (Proposed by W. HOPPS.)—In an acute-angled scalene triangle to inscribe the rectangle of minimum diagonal, having one of its sides coincident with a side of the triangle.

Solution by T. DOBSON, B.A.; R. TUCKER, M.A.; W. H. LAVERTY; *the* PROPOSER; *and many others.*



In the triangle ABC draw BD perpendicular to BC and equal to the perpendicular from A on BC, and join CD.

Draw *any* right line parallel to BC, cutting BD, CD in E and F, and AB, AC in G and H. Then

$$\frac{\triangle DEF}{\triangle DBC} = \frac{DE^2}{DB^2} = \frac{\triangle AGH}{\triangle ABC};$$

but $\triangle DBC = \triangle ABC$; $\therefore \triangle DEF = \triangle AGH$; and $\therefore EF = GH$.

Hence the rectangle on the diagonal BF, and the inscribed rectangle of which GH is a side, are identically equal; and it is obvious that BF is a minimum when at right angles to CD. Hence this

Construction.—Draw BF perpendicular to CD, and from F draw perpendiculars to BC, BD; then the rectangle on diagonal BF is evidently equal to the rectangle of minimum diagonal inscribed in ABC, and having a side in BC.

COR.—If $BC = a$, $BD = p_1$, $BF = u_1$; we have $u_1^{-2} = a^{-2} + p_1^{-2}$.

[From Mr. DOBSON'S *Corollary* we readily obtain

$$u_1^{-2} - u_2^{-2} = (b^{-2} - a^{-2}) \cot^2 C;$$

whence we see that the rectangle on the *longest* side of the triangle will have the *shortest* diagonal.]

2612. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—If points P, Q, R be taken at random, one on each of the sides BC, CA, AB, of the triangle ABC, the chance that the area of the triangle PQR shall be greater than $(n + \frac{1}{4})$ of the area of the triangle ABC (n being positive and $< \frac{3}{4}$) is

$$\frac{3-4n}{4} - \frac{12n+1}{4} \log \frac{4}{4n+1} + 8n^{\frac{3}{2}} \left(\tan^{-1} \frac{1}{2n^{\frac{1}{2}}} - \frac{\pi}{6} \right).$$

I. *Solution by the* PROPOSER.

Taking $BP = ax$, $CQ = by$, $AR = cz$, the area Δ' of the triangle PQR, in parts of the area of ABC, is

$$\Delta' = 1 - z(1-y) - x(1-z) - y(1-x) = 1 - x - y - z + yz + zx + xy,$$

or, putting $x = X + \frac{1}{2}$, &c., we have $\Delta' = \frac{1}{4} + YZ + ZX + XY$; hence the chance required is the chance that $YZ + ZX + XY$ shall be greater than n; X, Y, Z being restricted within the limits $\pm \frac{1}{2}$, and all values within those limits equally probable. This amounts to finding the volume common to a

hyperboloid of revolution of two sheets (when n is positive) and to a cube of volume unity whose diagonal coincides with the axis of the hyperboloid and whose faces are equally inclined to that axis. It is easily seen that this is six times the volume included within the hyperboloid and the planes $Z=X$, $Z=Y$, $Z=\frac{1}{2}$. But the area of hyperbola $(X+z)(Y+z)=z^2+n$ included between the straight lines $X=z$, $Y=z$, is $3z^2-n-(z^2+n)\log\frac{4z^2}{z^2+n}$, hence the volume required is $6\int_{(\frac{1}{2}n)^{\frac{1}{2}}}^{\frac{1}{2}}\left\{3z^2-n-(z^2+n)\log\frac{4z^2}{z^2+n}\right\}dz$,

the lower limit being the value of z at the vertex of the hyperboloid. The integration presents no difficulty, and gives the result as above.

Hence, if $n=0$, the chance that $\Delta'>\frac{1}{4}$ is $\frac{3}{4}-\frac{1}{2}\log 2$,
$n=\frac{1}{4}$, the chance that $\Delta'>\frac{1}{2}$ is $\frac{1}{2}-\log 2+\frac{1}{12}\pi$,
$n=\frac{3}{4}$, the chance that $\Delta'>1$ is 0, as it obviously ought to be.

[A Solution of the case when $n=\frac{1}{4}$ or $\Delta'>\frac{1}{2}$ was given by the EDITOR under Quest. 1230 in the *Educational Times* for October, 1862, and on p. 96 of Vol. II. of the *Reprint*.]

II. *Solution by* STEPHEN WATSON.

Let L, M, N be the middle points of BC, CA, AB, and put $PL=ax$, $MQ=by$, $NR=cz$; then we readily find that the area Δ' of the triangle PQR, in parts of that of ABC, is

$$\Delta'=1-(\tfrac{1}{2}-x)(\tfrac{1}{2}+z)-(\tfrac{1}{2}+x)(\tfrac{1}{2}-y)-(\tfrac{1}{2}+y)(\tfrac{1}{2}-z)$$
$$=\tfrac{1}{4}+xy+yz+zx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

When PQR is to be $>\frac{1}{4}$ ABC, or $\Delta'>\frac{1}{4}$, (2) becomes

$$xy+yz+zx>0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

which is fulfilled for all positive values of x, y, z; also for all negative values when x, y, z are *all* negative.

The chance in either case is $\frac{1}{8}$, and therefore of both $=\frac{1}{4}$ ..... (3).

When x is negative, or yz are *both* negative, in either case (2) gives $x<\frac{yz}{y+z}$; hence, in both cases, we have

$$2\int_0^{\frac{1}{2}}dy\int_0^{\frac{1}{2}}\frac{yz}{y+z}\,dz=\int_0^{\frac{1}{2}}\left(y-2y^2\log\frac{y+\frac{1}{2}}{y}\right)dy=\tfrac{1}{6}-\tfrac{1}{8}\log 2 \ldots (4).$$

The result is the same when y or zx is negative, and also when z or xy is negative, and these include all possible favourable cases; hence, putting p_1 for the probability in this case, we have

$$p_1=(3)+3(4)=\tfrac{3}{4}-\tfrac{1}{2}\log 2.$$

When PQR is to be $>\frac{1}{2}$ ABC, or $\Delta'>\frac{1}{2}$, (2) becomes

$$xy+yz+zx-\tfrac{1}{4}<0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5);$$

therefore $z>\frac{\frac{1}{4}-xy}{x+y}$; and since this must be $<\frac{1}{2}$, therefore $y>\frac{\frac{1}{2}-x}{1+2x}$.

Also it is plain from (6) that neither one nor two of the quantities x, y, z can be negative, but all three may; hence, doubling on account of this, and putting p_2 for the probability in this case, we have

$$p_2 = 2\int_0^{\frac{1}{2}} dx \int_{\frac{\frac{1}{2}-x}{1+2x}}^{\frac{1}{2}} \frac{\frac{1}{4}-xy}{x+y}\,dy$$

$$= 2\int_0^{\frac{1}{2}} \left\{x + (x^2+\tfrac{1}{4})\log\frac{x^2+\frac{1}{4}}{(x+\frac{1}{2})^2}\right\} dx$$

$$= \frac{1}{2} + \frac{\pi}{12} - \log 2.$$

2704. (Proposed by J. J. WALKER, M.A.)—A central conic $(a^2y^2 \pm b^2x^2 = a^2b^2)$ is turned about its centre in its own plane through a right angle; prove that the locus of the intersection of any tangent to the given conic with the same line in its new position is one of the two quartics

$$(x^2+y^2)^2 = a^2(x \mp y)^2 \pm b^2(x \pm y)^2.$$

I. *Quaternion Solution by* W. H. LAVERTY, B.A.

The vectors to the two tangents will evidently be

$$\rho = \quad \alpha\cos\tau + \beta\sin\tau + x(-\alpha\sin\tau + \beta\cos\tau),$$

and $$\rho' = -\alpha\sin\tau + \beta\cos\tau + y(\ \alpha\cos\tau + \beta\sin\tau).$$

At their intersection $\rho = \rho'$, and therefore $x = y = 1$, and we get for the equation to the locus

$$\rho = \alpha(\cos\tau - \sin\tau) + \beta(\sin\tau + \cos\tau);$$

therefore $\tau^2\rho = \tau^2\alpha(\cos\tau - \sin\tau)^2 + \tau^2\beta(\cos\tau + \sin\tau)^2$, since $S\alpha\beta = 0$;

therefore $\tau^4\rho = a^2(\tau\rho\,.\cos\tau - \tau\rho\,.\sin\tau)^2 + b^2(\tau\rho\,.\cos\tau + \tau\rho\,.\sin\tau)^2$;

or $$(x^2+y^2)^2 = a^2(x-y)^2 + b^2(x+y)^2.$$

Similarly for the hyperbola.

II. *Solution by the* PROPOSER; REV. J. L. KITCHIN; *and others.*

Taking b^2 positive, and supposing the direction of rotation to be from the positive side of the axis of x towards the negative side of the axis of y, if the equation to any given tangent be $a^2y'y + b^2x'x = a^2b^2$, that of the same line in its new position must be $a^2y'x - b^2x'y = a^2b^2$, since these two lines are at right angles to one another and cut off equal intercepts, measured from the centre, on *different* axes. The coordinates of the point of intersection of these lines are

$$(a^4y'^2 + b^4x'^2)x = a^2b^2(a^2y' + b^2x') \text{ and } (a^4y'^2 + b^4x'^2)y = a^2b^2(a^2y' - b^2x') \ldots(1).$$

Squaring and adding these equations, and dividing by $a^4y'^2 + b^4x'^2$,

$$(a^4y'^2 + b^4x'^2)(x^2+y^2) = 2a^4b^4 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Multiplying equations V.

$$\dots = x^{2^{n-1}} y^{2^{n-1}} - y^{2^{n-1}} x^{2^{n-1}} \dots$$

But $xy^{\dots} - y^{\dots} x^{\dots} = x^{\dots} y^{\dots} - y^{\dots} x^{\dots}$ and $xy^{2} - y^{2} x^{2} = 0$.

Hence [illegible] and [illegible] divided by [illegible] gives

$$x^{\dots} - y^{\dots} x^{\dots} = x^{\dots} y^{\dots} - y^{\dots} x^{\dots}.$$

Eliminating $x^{2^{n-1}}$ there remains for the required form

$$x^{\dots} - y^{\dots} x^{\dots} = \dots - y^{\dots} x^{\dots} + y^{\dots} (\dots$$

which, after reduction, becomes

$$x^{\dots} - y^{2^n} = x^{\dots} - y^{\dots} x^{\dots} + y^{2^n}.$$

If the equation had been in the oppositic direction, the equ[illegible] tangent in its new [illegible] would have been $Px + y - x'y'x =$ [illegible] sign if y would have had changed in the corresponding term.

Note on Anallagmatic Pavements. (Vol. xxii.);

by Professor Sylvester.

See diagram opposite.

In the Anallagmatic Pattern or Pavement here drawn (so called from relation of the lines or columns to one another remaining unaltered upon from one to another) it will be observed that each line or column consists of 6 squares of one colour and 10 of the contrary colour: there being lines or columns of the kind where black preponderates, and 8 where white preponderates.

In general, a pattern containing 2^{2n} (say n) compartments in each of lines and columns may be constructed not only anallagmatic, but so as to satisfy a similar condition to the above, viz., such that half of the lines and columns shall each contain $\frac{1}{2}(n + \sqrt{n})$ white and $\frac{1}{2}(n - \sqrt{n})$ black, while each of the remaining lines and columns contain $\frac{1}{2}(n - \sqrt{n})$ white and $\frac{1}{2}(n + \sqrt{n})$ black compartments. Pavements of this character, where the difference between the colours in each line and column is constant, may be termed isochromatic. It is easy to prove that anallagmatic pavements cannot be made isochromatic when n is an odd power of 2.

The construction of anallagmatic synchromatic squares it was which led the author to the discovery of the transcendental property of the binomial coefficients given in Question 2802 of the *Educational Times*.

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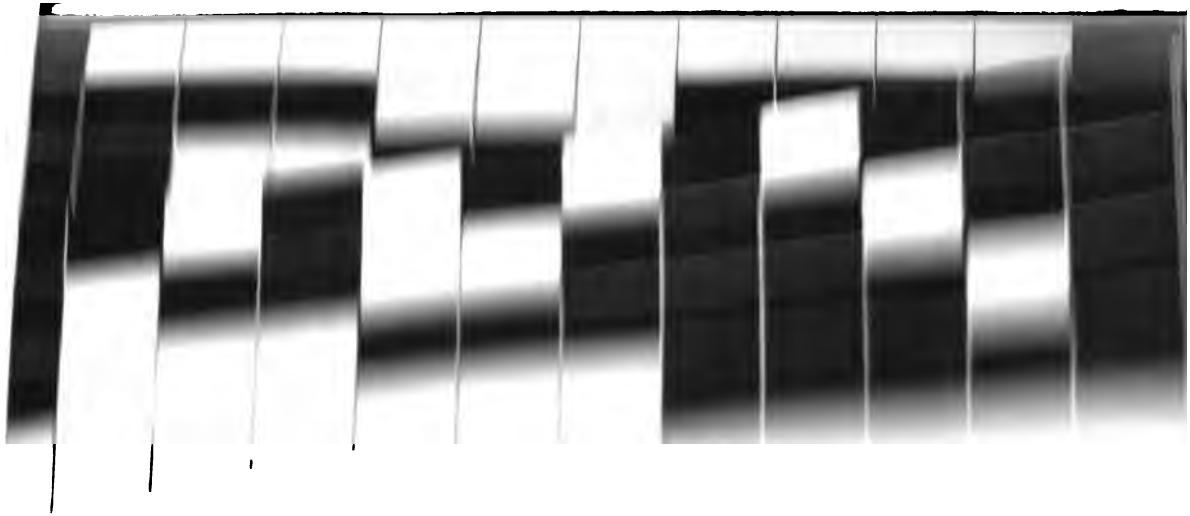

Multiplying equations (1),

$$(a^4y'^2 + b^4x'^2)^2 xy = a^4b^4(a^4y'^2 - b^4x'^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

But $a^4y'^2 + b^4x'^2 = b^2\{a^4 - (a^2 - b^2)x'^2\}$ and $a^4y'^2 - b^4x'^2 = b^2\{a^4 - (a^2 + b^2)x'^2\}$.

Hence (2) becomes $(a^2 - b^2)(x^2 + y^2)x'^2 = a^4(x^2 + y^2 - 2b^2)$,
and (3) divided by (2) gives

$$\{a^2(x-y)^2 + b^2(x+y)^2\}x'^2 = a^4(x-y)^2.$$

Eliminating x'^2, there results for the required locus

$$(a^2 - b^2)(x^2 + y^2)(x-y)^2 = (x^2 + y^2 - 2b^2)\{a^2(x-y)^2 + b^2(x+y)^2\},$$

which, after reduction, becomes

$$(x^2 + y^2)^2 = a^2(x-y)^2 + b^2(x+y)^2.$$

If the rotation had been in the opposite direction, the equation to the tangent in its new position would have been $b^2x'y - a^2y'x = a^2b^2$, and the sign of y would have been changed in the corresponding locus.

Note on Analagmatic Pavements, Quest. 2552; *by* Professor Sylvester.

(*See diagram opposite.*)

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EDITED BY

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(*a.*) The perpendiculars of a triangle meet in a point.

(*b.*) The middle points of the diagonals of a quadrilateral are in one straight line.

(*c.*) The circles whose diameters are the diagonals of a quadrilateral have a common radical axis.

(*d.*) Every rectangular hyperbola circumscribing a triangle passes through the intersection of perpendiculars.

(*e.*) Every rectangular hyperbola to which a triangle is self-conjugate passes through the centres of the four touching circles.

(*f.*) $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

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2494. On each side of a hexagon as base, a triangle is described by producing the two adjacent sides to meet, and a second hexagon is formed by joining the vertices of these triangles in order. Show that if either of these two hexagons can be inscribed or circumscribed to a conic, the other can be circumscribed or inscribed to a conic 44

2513. Let A, B, C, D, E, F be six points. Let BF, CD meet in *a*; AB, CE in *b*; AB, CD in *c*; AF, CE in *d*; BF, DE in *e*; and AF, DE in *f*. Show that, if either of the hexagons ABCDEF, *abcdef* can be inscribed or circumscribed to a conic, the other can be circumscribed or inscribed to a conic 44

2525. Three equidistant lines are drawn parallel to an asymptote of a hyperbola, and a triangle inscribed where the lines meet the curve. Then any line parallel to the other asymptote will be divided harmonically by the three sides of the triangle and the curve 20

2527. Find the locus of the intersection of two tangents to a circle, the chord of contact of which subtends a right angle at a fixed point; and also of the middle points of a system of chords of a circle, which subtend a right angle at a fixed point; and thence show that the envelope of a system of chords of a conic, which subtend a right angle at the focus, is another conic of coincident focus; and that the locus of the middle points of a system of chords of a conic which subtend a right angle at a fixed point is another conic 93

2533. Find the envelope of a line upon which two given conics intercept segments which have a common middle point; and find also the locus of this middle point..... 84

2539. If three conics cut one another so that every two of them have a common chord which is also a chord of a fourth conic, the other three chords of intersection of the three conics meet in a point 44

2544. There are two values of d for which α, β, γ, the roots of the cubic $ax^3 + 3bx^2 + 3cx + d = 0$ fulfil the linear relation $A\alpha + B\beta + C\gamma = 0$. Show how to find the quadratic equation of which these values of d are the roots 64

2597. A right cone, whose weight may be neglected, is suspended from a point in its rim; it contains as much fluid as it can: show that the whole pressure upon its surface is

$$\tfrac{1}{3}\pi\rho g h^4 \frac{\sin\alpha\cos\theta}{\cos^2\alpha}\left\{\frac{\cos(\theta+\alpha)}{\cos(\theta-\alpha)}\right\}^{\frac{3}{2}},$$

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where h and $2a$ are the height and vertical angle of the cone, and θ is determined from $3 \sin 2\theta = 4 \sin 2(\theta - a)$ 101

2598. A uniform rod (mass $= \mu$) is placed inside a spherical shell (mass $= m$, radius $= a$), which, under the influence of gravity, rolls down the surface of a rough sphere (radius $= r$); find equations for the movements of the rod and shell, there being no friction between them; and hence show that, if a particle μ is placed near the lowest point of a spherical shell (m) which performs small oscillations on a rough horizontal plane, the length of the simple equivalent pendulum is $\frac{2m}{2m+\mu} a$ 52

2603. In a plane, (1) on every line, there are two points harmonic conjugates to the conics through four points A, B, C, D. They are real, if the product of the four triangles ABC, BCD, CDA, DAB, and the perpendiculars from the points A, B, C, D on the line, is positive. If a line does not cut a conic, why are the conjugates on the line real, to any set of four points on the conic? Give a rule for determining the nature of the points on a line, when the product of the triangles is (1) positive, (2) negative.

(2.) Through any point, there are two lines conjugate to the conics inscribed in four lines. They are real in those regions of the plane formed by the four lines in which the product of perpendiculars from the point has the same sign as that of points in the convex quadrilateral. Hence every point on or within a conic inscribed in the four lines lies in these regions..... 105

2606. The developable circumscribing two surfaces of the second degree touches either of them along a curve, which is its intersection with another surface of the second degree..... 68

2616. Let two intersecting tetrahedra have all their edges bisected by the same system of Cartesian axes, each axis through two opposite edges of each tetrahedron; then the solid about the origin has the origin for its centre of figure..... 47

2618. A pack of n different cards is laid, face downwards, on a table. A person names a certain card, that and all the cards above it are shown to him and removed; he names another, and the process is repeated. Prove that the chance of his naming the top card during the operation is

$$1 - \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} - \frac{1}{\lfloor 4} + \ldots\ldots - \frac{(-1)^n}{\lfloor n} \quad \ldots\ldots 70$$

2622. The equation connecting the distances (r_1, r_2, r_3) of any point on a Cartesian oval from the foci is

$$(\beta - \gamma) a^{\frac{1}{2}} r_1 + (\gamma - \alpha) \beta^{\frac{1}{2}} r_2 + (\alpha - \beta) \gamma^{\frac{1}{2}} r_3 = 0,$$

where α, β, γ are the distances of the foci from the triple focus 56

2623. If a conic pass through the four points of contact of tangents to a cubic from a point (A) on the curve, and through two other points (B, C) on the cubic; then A is the pole of BC with regard to the conic 68

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2636. 1. If A, B be two confocal conics, and A′ be a third conic having double contact with A, show that there exists a fourth conic B′ having double contact with B and confocal with A′.

2. If A, B be two confocal quadrics, and A′ be a third quadric having continuous contact with A, show that there exists a fourth quadric B′ having continuous contact with B and confocal with A′ 43

2640. 1. If a conic of given eccentricity e circumscribe a triangle ABC, the locus of its centre is

$$\frac{(yz\sin^2 A + zx\sin^2 B + xy\sin^2 C)^2}{xyz\,(x+y+z)\sin^2 A\sin^2 B\sin^2 C} = \frac{(e^2-2)^2}{1-e^2};$$

and (2) if it be inscribed in the triangle, the locus of the centre is

$$\frac{(x^2\sin^2 A + \ldots + 2yz\sin B\sin C\cos A + \ldots)^2}{xyz\,(x+y+z)\sin^2 A\sin^2 B\sin^2 C} = \frac{4(e^2-2)^2}{1-e^2};$$

the triangle of reference having its angular points at the middle points of the sides of the triangle ABC.

3. Hence show that the locus of points with which as centre two similar conics can be drawn, one inscribed and the other circumscribed to the triangle, is two circles, one the circumscribed circle of the triangle, and the other that which has the centre of gravity and the centre of perpendiculars of the triangle as the ends of a diameter 25

2642. If three surfaces of the second degree meet in four points, at each of which their three curves of intersection have a common tangent, these four points lie in one plane 77

2651. Determine the form of the solution of the differential equation

$$2f(x)\frac{d^2U}{du^2} + 3f'(x)\frac{dU}{du} + \left\{f''(x) \pm n^2\right\}U = 0 \ldots\ldots 103$$

2653. 1. If the centre of one of the four circles which touch the sides of a triangle self-conjugate with respect to a parabola lies on the directrix; show that its circumference passes through the focus.

2. If the point of intersection of the three perpendiculars of a triangle inscribed in a parabola lies on the directrix; show that the self-conjugate circle of this triangle passes through the focus of the curve 32

2654. Determine the polar reciprocal of the quartic curve

$$y^2z^2 + z^2x^2 + x^2y^2 + 2xyz\,(lx + my + nz) = 0 \ldots\ldots 60$$

2670. An observer, seated in the aisle of a cathedral and looking westward, sees the horizontal lines above the arches converging to a point before him, and consequently cutting the horizon at a certain angle. On his facing northwards, he sees that the same lines are now parallel to the horizon. Required the curve that they will appear to lie in, as he gradually turns his head through 90° 57

2688. A variable triangle circumscribes an equilateral hyperbola, and is such that its *nine-point* circle passes through the centre of this curve; prove that the locus of the centre of its *circumscribing* circle is the hyperbola in question 59

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2701. In a right conoid whose axis is the axis of z, show that the radii of principal curvature at the point ($r\cos\theta$, $r\sin\theta$, z) are given by the equation

$$4\rho^2\left(\frac{dz}{d\theta}\right)^2 + r\rho\frac{d^2z}{d\theta^2}\left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^{\frac{1}{2}} = \left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^2 \ldots\ldots 111$$

2712. Form 15 symbols into sets, 5 symbols in each set, so that every combination of 4 symbols shall appear once in the sets 97

2718. Find *in plano* the locus of a point P, such that from it two given points A, C, and two given points B, D, subtend equal angles 33

2719. 1. The six points which, in conjunction with any common transversal, divide harmonically the six sides of any tetrastigm, lie all on one conic, which also passes through the three points of intersection of opposite sides of the tetrastigm.

2. Of the six points, if those pairs which lie on opposite sides of the tetrastigm be joined by straight lines, the three straight lines thus drawn are concurrent, and their point of intersection is the pole of the transversal with respect to the conic 21

2724. Three smooth rings, P, Q, R, are thread on an endless string of given length, and constrained to move on three straight rods, OP, OQ, OR. To investigate the motion corresponding to a slight arbitrary disturbance of the system 75

2726. If a conic touch the sides of a triangle and pass through the centre of the circumscribed circle, the director circle of the conic will touch the circumscribed circle of the triangle 60

2727. Two lines (a) and (b) in space touch a fixed quadric U, show that a variable line intersecting the two former and touching the latter generates another quadric..... 20

2728. Given three planes and their poles with regard to a system of quadrics, the locus of centre is a right line 101

2736. Find the curve whose radius of curvature and radius vector at each point are equal to one another, and prove that it is the second involute to a circle having an apse midway between the cusp of the first involute (from which it is derived) and the centre of the circle..... 18

2740. Prove that the envelope of a line given by an equation of the form $\mathrm{L}\cos 2\phi + \mathrm{M}\sin 2\phi + \mathrm{P}\cos\phi + \mathrm{Q}\sin\phi + \mathrm{R} = 0$ may be obtained as the discriminant of a cubic equation, and form this equation 42

2741. A moveable event (depending on the moon) happened in the year y; show that it will happen again in the year $y + 19(28t - 5a + 8b)$; where a may be 0, 1, 2, 3; b the number of completed centuries since the year y which are not leap years; and t any arbitrary integer 52

2745. Show that the average area of all the ellipses that can be inscribed symmetrically in a portion of a parabola cut off by a double ordinate perpendicular to the axis, in parts of the area of the parabola, is $\frac{2^3}{1\,6\,0}\pi$ if (1) the centres be equably distributed over all possible positions on the axis, and $\frac{1}{6}\frac{1}{4}\pi$ if (2)

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the intersections of the chords of contact with the axis be equably distributed on the axis 24

2755. Form 16 symbols into sets, 4 symbols in each set, so that every triad in the symbols shall appear once in the sets 97

2757. If $x_0^2+y_0^2=1$, $x_1^2+y_1^2=1$, and $\begin{vmatrix} x, & y, & 1 \\ x_0, & y_0, & 1 \\ x_1, & y_1, & 1 \end{vmatrix} = L$;

show that each of the equations

$$\frac{a^2(x-x_0)^2+b^2(y-y_0)^2}{(xx_0+yy_0-1)^2} = \frac{a^2(x-x_1)^2+b^2(y-y_1)^2}{(xx_1+yy_1-1)^2} \quad \ldots\ldots(1),$$

$$\frac{a^2(x-x_0)^2+b^2(y-y_0)^2}{(xy_0-x_0y)^2-(x-x_0)^2-(y-y_0)^2} = \frac{a^2(x-x_1)^2+b^2(y-y_1)^2}{(xy_1-x_1y)^2-(x-x_1)^2-(y-y_1)^2} \quad \ldots(2),$$

represents the right line $L=0$ and a cubic curve 33

2761. Find the conic envelope of the radical axis of two circles, one of which is fixed, whilst the other passes through a fixed point. If the radii of the circles are equal, find the foci and axes of the envelope..... 74

2762. Find the envelope of an ellipse which has one vertex in the curve of a given parabola, and touches at its adjacent vertex the extremity of a fixed double-ordinate of the parabola 58

2763. Give a geometrical interpretation for the relation between the invariants of two conics, viz., $\Theta\Theta' = \Delta\Delta'$ 94

2764. It is well known that if A, P, P′ be fixed points on a line (that is, if the distances AP, PP′ are constant), and if PP′ describe given fixed lines, say the directrices, then the point A describes an ellipse having for its centre the intersection of the two directrices; and the question hence arises: Given three positions of the point A and of the generating line APP′, to find the directrices; or, what is the same thing, Given the lines 1, 2, 3, passing through the points A, B, C respectively, it is required to draw a line cutting the three lines in the points P, Q, R respectively, such that $AP=BQ=CR$..... 19

2766. Three points move from a fixed point in three different but given directions, with given uniform but different velocities. Find the locus of the centre of the circle passing through them 22

2769. From a point, taken at random within a triangle, perpendiculars are drawn on the sides; find the probability that the triangle formed by joining the feet of these perpendiculars is acute-angled 94

2770. If a conic be drawn through S, S′, the foci (both real or both impossible) of a given conic, and osculate the conic at a point P, the tangents at S, S′ will intersect in the centre of curvature at P 60

2771. Prove that (1) if a straight line of length a be divided at random in two points, the mean value of the sum of the squares on the three parts is $\frac{1}{3}a^2$; (2) if the line be divided at random into two parts, and the longer part again divided at random into two parts, the mean value of the sum of the

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squares on the three parts is $\frac{2}{7}\frac{5}{2}a^2$; and (3) if it is an even chance that n times the sum of the squares on the parts in (1) is less than the square on the whole line, then

$$n = \frac{12\pi}{4\pi + 3\sqrt{3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 78$$

2773. When a given angle rolls upon a fixed parabola, the locus of its vertex is well known to be a hyperbola having the same focus and directrix as the parabola. Conversely, when a given parabola rolls within a fixed angle, show that its focus and vertex describe lines of the fourth and sixth orders respectively; find the *actual equations* of both these curves, and thence show that the vertex of the fixed angle is a conjugate point on the former curve, and that the equation of the latter, when the given angle is a right angle, becomes

$$x^2y^2(x^2+y^2+3a^2) = a^6 \quad \text{or} \quad x^{\frac{2}{3}}y^{\frac{2}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}}) = a^2,$$

the sides of the given angle being the axes, and a equal to the distance of the focus from the vertex 95

2774. A central conic $(b^2x^2 \pm a^2y^2 = a^2b^2)$ is turned in its own plane about its centre through a right angle; prove that the locus of the intersection of the normal at any point on the given conic with the same line in its new position is one of the two sextic curves

$$(x^2+y^2)^2\left\{a^2(x \pm y)^2 \pm b^2(x \mp y)^2\right\} = (a^2 \mp b^2)^2(x^2-y^2)^2 \ldots\ldots \quad 96$$

2776. Through A, the double point of a circular cubic, draw AB perpendicular to the asymptote; if chords be drawn to the curve subtending a right angle at the double point, show that there is a fixed point in AB at which also they subtend a right angle 63

2778. Suppose μ any quantity fractional or integral, and write

$$c_0 = a_0;\ c_1 = \mu a_1,\ c_2 = \frac{\mu(\mu+1)}{2}a_2,\ c_3 = \frac{\mu(\mu+1)(\mu+2)}{2.3}a_3, \ldots\ldots;$$

also let n be a positive integer such that μ and $\mu+n$ have not different signs; then it has been proved elsewhere, and may be assumed to be true, that the number of changes of sign in the series

$$a_0^2,\ a_1^2 - a_0a_2,\ a_2^2 - a_1a_3,\ \ldots\ldots$$

is *an inferior limit* to the number of imaginary roots in the equation

$$c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \ldots\ldots = 0.$$

1. Required to prove that the above theorem remains true when we write

$$c_0 = a_0,\ c_1 = \frac{\mu}{\nu}a_1,\ c_2 = \frac{\mu(\mu+1)}{\nu(\nu+1)}a_2,\ c_3 = \frac{\mu(\mu+1)(\mu+2)}{\nu(\nu+1)(\nu+2)}a_3, \ldots\ldots,$$

provided ν is a positive integer, and $\mu-\nu+1$, $\mu+\nu$ do not differ in sign.

2. Ascertain whether ν being an integer is essential to the truth of the extended theorem 38

2779. Show that the famous equation between *arc, angle of con-*

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tingence, and *perpendicular on tangent* to a curve, viz., $p + \frac{d^2p}{d\phi^2} = \frac{ds}{d\phi}$, may be interpreted as affirming the *self-evident proposition* that the radius of curvature is the distance between the tangent at any point of a curve and the tangent at the corresponding point of its second evolute; and account for the connecting sign between p and $\frac{d^2p}{d\phi^2}$ in the above equation being *plus* and not *minus* 21

2780. The envelope of the chord common to an ellipse and its circle of curvature is a curve of the fourth class which has three double tangents, one at infinity and the two others coincident with the conjugate diameters equally inclined to the axis. Prove this, examine the curve, and consider the cases of the hyperbola and parabola 62

2781. Prove that the maximum value of the common chord of an ellipse and its circle of curvature, a and b being the semiaxes of the ellipse, is

$$\frac{4}{3\sqrt{3}(a^2-b^2)}\left\{2(a^4-a^2b^2+b^4)^{\frac{3}{2}}-(a^2+b^2)(2a^2-b^2)(2b^2-a^2)\right\}^{\frac{1}{2}}$$ 28

2782. 1. A given point is known to be within a certain circle of given radius, but unknown position; find the chance that another given point is also within the circle.

2. Three given points are known to be within a certain circle, which is otherwise altogether unknown; determine the most probable position of its centre.

3. Two given points are known to be within a circle, and a third given point is known to be outside it; determine the most probable position of its centre 107

2787. The value of $\iiint x^2 dx\, dy\, dz$ taken over the whole volume of a tetrahedron through one of whose corners the plane of (yz) passes, is $\frac{1}{5}V(a_1^2+b_1^2+c_1^2)$, where V is the volume, and a_1, b_1, c_1 are the distances of the middle points of the opposite edges from the plane of reference 110

2789. If two circles pass through the foci of an ellipse and touch the same variable tangent to the curve, the angle at which they intersect is constant and equal to $2\tan^{-1}\frac{b}{c}$ 62

2795. Find the average square of the distance between the centres of the inscribed and circumscribed circles of a triangle inscribed in a given circle 54

2797. In any complete quadrilateral, the radical axis of the three circles whose diameters are the diagonals will pass through the centres of perpendiculars of the four triangles formed by the four straight lines 55

2798. To find the condition that a given point (x, y, z) may be exterior to the given conic

$$S \equiv (A, B, C, F, G, H)(x, y, z)^2 = 0$$ 41

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2799. If n dice are thrown, what is the chance of an odd number of aces turning up? 61

2800. Three circles passing through a point P form a circular triangle ABC, and each side of this circular triangle or its continuation is cut orthogonally at the point P′ by a circle passing through P; prove that the three circles described about the triangles PAP′, PBP″, PCP‴ are coaxal..... 59

2804. A straight pole stands vertically on a slope inclined to the south. If it be broken at random by the wind blowing in a given direction, so that the upper end of the pole rests upon the slope, determine the probable area of the triangle thus formed; and deduce the result for a horizontal plane 69

2808. Four fixed tangents are drawn to a conic S; three other conics are drawn osculating S in any the same point P, and each passing through the ends of a diagonal of the circumscribed quadrilateral: prove that the tangents to these conics at the ends of the diagonal meet in one point P′, and that the locus of P′ is a curve of the sixth degree and fourth class, having two cusps on every diagonal, and touching S at the points of contact of the four tangents 108

2813. The equation of a conic referred to an axis and tangent at vertex being $ax^2 + by^2 + 2dx = 0$, if the conic be turned about the vertex in its own plane through a right angle, the locus of the intersection of any tangent in the original position with the same line in the new position of the conic is the (bicircular) quartic

$$b\left\{a(x^2+y^2)+2d(x+y)\right\}(x^2+y^2) = d^2(x-y)^2,$$

and the corresponding locus for the normal is the sextic

$$b\left\{a(x^2+y^2)+b(x-y)^2\right\}\left\{a(x^2+y^2)+d(x+y)\right\}^2 = (a-b)^2d^2(x^2-y^2)^2.$$

What does this latter equation become in the case of the parabola? 71

2815. A cask contains a gallons of wine. Through a hole in the top water or wine can be let in at the rate of b gallons per minute; and through a pipe in the bottom, when open, the mixture can escape at the same rate. Suppose the discharge pipe is opened at the same instant that water is let in at the top, and t minutes afterwards the water is shut off and wine let in. Required the quantity of water in the cask at the end of t_1 minutes from the opening of the discharge pipe, and the length of time elapsed, both before and after wine was let in at the top, when the quantities of the two fluids in the cask were equal, supposing them to mingle perfectly..... 87

2816. Find three square numbers in arithmetical progression, such that the square root of each (α) increased or else (β) diminished by unity shall give three rational squares 88

2818. Given a circle S and a straight line α not meeting S in real points; O, O′ are the two point-circles to which, and S, α is the radical axis; two conics are drawn osculating S in the same point P, and having one focus at O, O′ respectively: prove that the corresponding directrices coincide 74

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2821. Show that the mean value of the distance from one of the foci of all points within a given prolate spheroid is $\frac{1}{4}a(3+e^2)$, $2a$ being the axis and e the eccentricity 87

2823. Show that on a chess-board the chance of a rook moving from one square to another without changing colour is $\frac{8}{7}$; but that without altering the equality of the *number* of the black and white squares, but only the manner of their distribution, the chance may be made equal to $\frac{1}{2}$ 49

2828. A straight rod is divided into n parts in arithmetical progression, and equal particles are fixed at the points of division. If the system be made to vibrate about one extremity, determine the average length of the pendulum, neglecting the weight of the rod 83

2835. In a plane, if A, B are two points, and a point P describe a curve of the nth order, show that P′, the intersection of the perpendiculars in the triangle PAB, will describe a curve of the $2n$th order with three multiple points of the order n. Explain why the curve corresponding to a circle through the points A and B includes its reflexion to the line AB 85

2839. In a triangle the bisector of the base is equal to the less side and also to one-half of the greater side; determine the three angles 73

2847. Prove that

$$\epsilon^{hD^2}.\epsilon^{-kx^2} = (1+4hk)^{-\frac{1}{2}}\epsilon^{-\frac{kx^2}{1+4hk}}, \text{ where } D = \frac{d}{dx} \text{ .....} \quad 67$$

2870. Given three points A, B, C, and a conic; two points P, Q are taken on the conic such that the pencil $A\{BPQC\}$ is harmonic; prove that the envelope of PQ is a conic touching AB, AC at points on the polar of A with respect to the given conic 83

2875. Any tangent to a conic is, of course, divided in involution by three other tangents, and the lines joining their points of intersection to one of the foci of the conic; prove that the distance between the double points of the involution subtends a right angle at the focus of the conic. The locus of these double points, the three tangents being fixed, is a cubic having a double point at the focus, whose nature I have not examined 106

2877. If

$$u = x+y+z+t, \qquad v = fx+gy+hz+kt,$$
$$U = yzt+ztx+txy+xyz, \qquad V = fyzt+gztx+htxy+kxyz;$$

prove that the resultant of u, v, U, V is

$$(f-g)(f-h)(f-k)(g-h)(g-k)(h-k)$$
$$\times (f+g-h-k)^2(f+h-g-k)^2(f+k-g-h)^2 \text{ .....} \quad 81$$

2887. 1. If a point P on a conic be connected with any two fixed points A and B in its plane, all chords which are divided harmonically by PA and PB will be concurrent.

2. The locus of the point of concurrence when P is variable

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is another conic which has double contact with the given one on AB 100

2893. If there be (n) quantities a, b, c, d, &c., each of which takes independently a given number of values $a_1a_2a_3\ldots$ $b_1b_2b_3\ldots$ &c. (the number may be different for each), if we put

$$\Sigma(a) = a + b + c + d + \&c.,$$

and if for shortness we denote "the mean value of x" by $M(x)$, prove that

$$M[\Sigma(a)] = M(a) + M(b) + M(c) + \&c. = \Sigma[M(a)],$$

$$M[\Sigma(a)]^2 = \{\Sigma[M(a)]\}^2 - \Sigma[M(a)]^2 + \Sigma[M(a^2)]. \ldots\ldots 99$$

2895. At any point on a cusped cubic a tangent is drawn meeting the cubic in a second point; from the first point a line is drawn touching the cubic in a third point. Prove that the lines drawn from the cusp to these points form with the cuspidal tangent a harmonic pencil 104

2898. A circle rolls with internal contact on a fixed circle of half its radius; prove that the envelope of any chord of the rolling circle is a circle which reduces to a point for a diameter 105

2902. The circle passing through three points on a parabola, the normals at which co-intersect, always passes through the vertex; and if the point of co-intersection of the normals describe a coaxial conic having the vertex as centre and axes in the ratio of $m : n$, the locus of the centre of the circle will be a conic having the focus as centre, and axes in the ratio of $2m : n$ 104

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

1878. (Proposed by W. K. CLIFFORD, B.A.)—If a line of given length be marked at random in n points, and broken up at those points, find the chance that the sum of the squares on the parts shall not exceed one-nth of the square on the whole line.

Solution by the Rev. J. WOLSTENHOLME, M.A.

Let $x_1, x_2, \ldots x_n$ be the distances, in order of magnitude, of the n points of division from one extremity, and 1 the length of the line; then the limiting equation is

$$x_1^2 + (x_2 - x_1)^2 + \ldots (1 - x_n)^2 < \frac{1}{n},$$

or
$$x_1^2 + \ldots + x_n^2 - x_1 x_2 - \ldots - x_{n-1} x_n + \frac{n+1}{2n} < 0.$$

Now I assume as obvious that a point (in space of n dimensions) whose coordinates (rectangular or not) are $x_1, x_2, \ldots x_n$, is a random point when $x_1, x_2, \ldots x_n$ are random lengths, and that the chance required is the value of $\iint \ldots dx_1 dx_2 \ldots dx_n$ when the limits are given by the equation

$$x_1^2 + \ldots + x_n^2 - x_1 x_2 - \ldots + \frac{n+1}{2n} = 0 \ldots\ldots\ldots\ldots (A),$$

divided by the same integral when the limits are given by $x_1 < x_2 < x_3 \ldots < x_n < 1$, all positive.

First, to find the quasi-volume of the quadric (A). Its quasi-centre is given by

$$2x_1 - x_2 = 2x_2 - x_1 - x_3 = 2x_3 - x_2 - x_4 = \ldots = 2x_n - x_{n-1} - 1 = 0,$$

or $x_2 = 2x_1, \quad x_3 = 3x_1 \ldots,$ or $x_1 = \frac{1}{n+1}, \quad x_2 = \frac{2}{n+1} \ldots$

The equation of the surface referred to its centre is then

$$x_1^2 + \ldots + x_n^2 - x_1 x_2 - \ldots - x_{n-1} x_n = \frac{1}{2n(n+1)}.$$

Now, transforming the axes (and to facilitate matters here I assume rectangular axes) so as to destroy the terms involving products, the resulting equation is

$$a_1 X_1^2 + a_2 X_2^2 + \ldots + a_n X_n^2 = \frac{1}{n(n+1)},$$

where (see my Book of Mathematical Problems, Quest. 921) $a_1, a_2, \ldots a_n$ are the roots of the equation

$$\begin{vmatrix} z-2, & 1, & 0, & 0, & \ldots\ldots\ldots\ldots\ldots\ldots \\ 1, & z-2, & 1, & 0, & 0, \ldots\ldots\ldots\ldots\ldots \\ 0, & 1, & z-2, & 1, & 0, \ldots\ldots\ldots\ldots\ldots \\ \vdots & & \vdots & & \vdots \\ 0, & 0, & 0, & \ldots\ldots\ldots\ldots & 1, \quad z-2, \quad 1 \\ 0, & 0, & 0, & \ldots\ldots\ldots\ldots & 0, \quad 1, \quad z-2 \end{vmatrix} = 0,$$

the discriminant of

$$(x_1^2 + \ldots + x_n^2) z - 2 (x_1^2 + \ldots + x_n^2 - x_1 x_2 - \ldots),$$

and the product of these roots is $n+1$. Also the value of $\iint\ldots dX_1 \ldots dX_n$, where the limits are given by $a_1 X_1^2 + \ldots + a_n X_n^2 = \frac{1}{n(n+1)}$ is (Todhunter's *Integral Calculus*, Art. 274)

$$\frac{1}{(a_1 a_2 \ldots a_n)^{\frac{1}{2}}} \left\{\frac{\pi}{n(n+1)}\right\}^{\frac{1}{2}n} \frac{1}{\Gamma(\frac{1}{2}n+1)} = \frac{1}{(n+1)^{\frac{1}{2}}} \left\{\frac{\pi}{n(n+1)}\right\}^{\frac{1}{2}n} \frac{1}{\Gamma(\frac{1}{2}n+1)};$$

and $\iint\ldots dx_1 dx_2 \ldots dx_n$, over all such positive values that $x_1 < x_2 < x_3 \ldots < x_n < 1$, is $\frac{1}{\lfloor n}$ or $\frac{1}{\Gamma(n+1)}$. The chance required is then

$$\frac{1}{(n+1)^{\frac{1}{2}}} \left\{\frac{\pi}{n(n+1)}\right\}^{\frac{1}{2}n} \frac{\Gamma(n+1)}{\Gamma(\frac{1}{2}n+1)}.$$

[The Proposer's solution is given on p. 83 of Vol. VI. of the *Reprint*.]

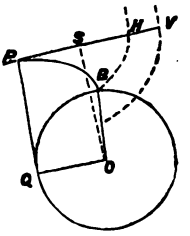
2736. (Proposed by Professor SYLVESTER.)—Find the curve whose radius of curvature and radius vector at each point are equal to one another, and prove that it is the second involute to a circle having an apse midway between the cusp of the first involute (from which it is derived) and the centre of the circle.

Solution by the REV. JAMES WHITE, M.A.

The property of the second involute of a circle stated in this Question, viz., that when its apse is midway between the cusp of the first involute (from

which it is derived) and the centre of the circle, its radius vector and radius of curvature will be equal, may be proved geometrically, as follows :—

Let V be any point on this second involute whose apse is midway between O and B: it is required to prove that OV = PV.



The second involute which commences at B, the cusp of the first, will cut PV in H, making $HV = \frac{1}{2}a$, (a being radius of the circle). Let $QP = t$; and draw OS perpendicular to VP; then $OS = t$, and $SP = a$.

Now $OV^2 = OS^2 + VS^2 = t^2 + (VP - a)^2$,

and $$VP = HP + \tfrac{1}{2}a = \text{arc } BP + \tfrac{1}{2}a = \frac{t^2}{2a} + \frac{a}{2};$$

consequently $$OV^2 = t^2 + \left(\frac{t^2}{2a} - \frac{a}{2}\right)^2 = \left(\frac{t^2}{2a} + \frac{a}{2}\right)^2;$$

therefore $$OV = \frac{t^2}{2a} + \frac{a}{2} = VP;$$

that is, radius vector (OV) = radius of curvature (VP).

[Mr. WHITE states that the main part of this proof is due to Mr. CROFTON.]

From this a simple method of finding a point on the second involute corresponding to any one given on the first may be derived.

Bisect the line joining the given point with the centre of the circle, and at the point of bisection erect a perpendicular. Also at the given point erect a perpendicular to the tangent from it to the circle; the intersection of these perpendiculars is a point on the second involute which has its apse midway between the cusp of the first and the centre of the circle (the triangle OVP being isosceles).

Take a point (H) on the perpendicular to the tangent distant from the one found one-half the length the radius of the circle, and it is a point on the second involute which commences at the cusp of the first.

[Professor SYLVESTER sends the following remarks on this question :—

The envelope of a set of half-pitch second involutes to any given circle is a circle concentric with the first. Consequently *any* circle corresponds to a singular solution of the differential equation which defines the curve in question; and thus we obtain another and obviously true solution of the question.]

2764. (Problem by A. SMITH, Q.C.)—It is well known that if A, P, P′ be fixed points on a line (that is, if the distances AP, PP′ are constant), and if P, P′ describe given fixed lines, say the directrices, then the point A describes an ellipse having for its centre the intersection of the two directrices; and the question hence arises: Given three positions of the point A and of the generating line APP′, to find the directrices; or, what is the same thing, Given the lines 1, 2, 3, passing through the points A, B, C respectively, it is required to draw a line cutting the three lines in the points P, Q, R respectively, such that AP = BQ = CR.

Solution by ARCHER STANLEY.

Let p, q, r be any three corresponding points on the three given lines; that is to say, let the three segments Ap, Bq, Cr (set off in assigned directions) be equal to one another. Then the connector pq will, by a well-known theorem, envelope a parabola of which Ap is a tangent; in like manner, the connector pr will envelope another parabola having the tangent Ap in common with the first. The remaining two common tangents of these parabolas will manifestly be the two required directrices. The problem is thus reduced to the well-known one (soluble by rule and compass) of finding the common tangents to two parabolas which touch a given line.

2525. (Proposed by G. O. HANLON.)—Three equidistant lines are drawn parallel to an asymptote of a hyperbola, and a triangle inscribed where the lines meet the curve. Then any line parallel to the other asymptote will be divided harmonically by the three sides of the triangle and the curve.

Solution by the Rev. J. WOLSTENHOLME, M.A.; *the* PROPOSER; *and others.*

First to find the locus of a point on any line in a given direction forming a harmonic range with the points where it meets the sides of the triangle of reference.

If $(l+k)x+(m+k)y+(n+k)z=0$ be any such straight line, then for the point (P) in question we have $(m+k)y=(n+k)z$ (if A'B'C'P be harmonic), and the locus is

$$(lx+my+nz)(y-z)+(x+y+z)(nz-my)=0 \quad \ldots\ldots\ldots(1);$$

or, in another form, $2(m-n)yz-(n-l)zx-(l-m)xy=0$ ..... (2).

Equation (2) shows that this is a conic circumscribing the triangle, and (1) that the asymptotes are parallel to $lx+my+nz=0$ and to $y=z$; the latter of which being the bisector of the side BC, shows that the three straight lines through A, B, C parallel to one asymptote are equidistant. Hence the truth of the theorem.

2727. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—Two lines (a) and (b) in space touch a fixed quadric U, show that a variable line intersecting the two former and touching the latter generates another quadric.

Solution by W. S. BURNSIDE, M.A.

It will be convenient to write the equation of the quadric U under the form $ax^2+by^2+2nxy+2rz\omega=0$, where the planes z and ω plainly touch the surface, and the line (x, y) is their chord of contact. Whence we may consider (x, z) and (y, ω) as the tangent lines (a) and (b). This being so, the condition that a variable line $x=\lambda z$, $\omega=\mu y$, should touch the quadric U is $ab\lambda^2=(n\lambda+\mu r)^2$, giving for the surface traced out by it the equation $abx^2y^2=(nxy+rz\omega)^2$, which resolves itself into the two constituents $(n\pm\sqrt{ab})xy+rz\omega=0$, which represent two quadrics.

2779. (Proposed by Professor SYLVESTER.) — Show that the famous equation between *arc, angle of contingence,* and *perpendicular on tangent* to a curve, viz., $p + \frac{d^2p}{d\phi^2} = \frac{ds}{d\phi}$, may be interpreted as affirming the *self-evident proposition* that the radius of curvature is the distance between the tangent at any point of a curve and the tangent at the corresponding point of its second evolute; and account for the connecting sign between p and $\frac{d^2p}{d\phi^2}$ in the above equation being *plus* and not *minus*.

Solution by J. J. WALKER, M.A.

1. Let (r, p, ϕ) refer to the given curve; and let (r_1, p_1, ϕ_1), (r_2, p_2, ϕ_2) refer similarly to its first and second evolutes respectively. Then it is evident that

$$p + p_2 = \rho,$$

where $\rho = \frac{ds}{d\phi}$ is the radius of curvature of the given curve.

Now $$p_1 = \frac{dp}{d\phi}, \quad \text{and} \quad p_2 = \frac{dp_1}{d\phi_1} = \frac{dp_1}{d\phi} = \frac{d^2p}{d\phi^2},$$

whence $$p + \frac{d^2p}{d\phi^2} = \rho = \frac{ds}{d\phi}.$$

2. In what precedes, the case has been considered of $\rho > p$. But if $\rho < p$, *i.e.*, $p - p_2 = \rho$, as p increases p_1 decreases, as is evident on drawing the figure; so that $p_2 = -\frac{dp_1}{d\phi_1} = -\frac{d^2p}{d\phi^2}$. Hence the sign between p and $\frac{d^2p}{d\phi^2}$ is in both cases *plus*.

2719. (Proposed by Professor WHITWORTH.)—1. The six points which, in conjunction with any common transversal, divide harmonically the six sides of any tetrastigm, lie all on one conic, which also passes through the three points of intersection of opposite sides of the tetrastigm.

2. Of the six points, if those pairs which lie on opposite sides of the tetrastigm be joined by straight lines, the three straight lines thus drawn are concurrent, and their point of intersection is the pole of the transversal with respect to the conic.

I. *Solution by* W. S. MCCAY, B.A.

It is not hard to prove this directly by Pascal's theorem; but it is easier to consider the line at infinity, then we know (1) that the locus of centres of conics through the four points is a conic through the six middle points of the lines of the system, and that three other points on the locus are the three intersections of opposite lines of the system, and (2) that the centre of this conic is the intersection of the three lines joining the middle points of opposite lines; for these three lines meet and bisect each other in the centre of gravity of equal weights at the four vertices of the tetrastigm.

II. *Solution by* F. D. THOMSON, M.A.

1. This follows at once from the well known theorem, that if a system of conics be described about a given quadrilateral, the locus of the pole of a fixed line with respect to these conics is another conic.

This theorem may be proved analytically, as in Salmon; or geometrically, as follows:

The given transversal is cut in points in involution by any conic of the system, and by the opposite sides of the quadrilateral; and the *double points* of the involution are the points where two conics of the system *touch* the transversal.

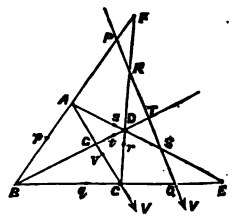
Now, for each of the points where the transversal cuts the required locus, the pole lies on its polar, and therefore must be the point of contact of the transversal with a conic of the system.

Hence the *only points* in which the transversal meets the required locus are the two double points of the involution determined by the intersections of the transversal and the opposite sides and diagonals of the quadrangle.

Hence the required locus is a *conic section*, and it is easily seen to pass through the points mentioned in the Question.

2. The former part of this proposition may be derived by projection from well known properties of the *middle* points of the sides of a quadrilateral; or *directly*, as follows:

Let the transversal meet the sides in P, Q, R, S, T, V; and let the points of harmonic section be p, q, r, s, t, v. Then



$$\{BpAP\} = \{DsAS\};$$

therefore $$T\{BpAP\} = T\{DsAS\}$$

$$= T\{BsAP\};$$

therefore Tps is a straight line. Similarly, Tqr, Vrs, Vqp, Rsv, Qpv, &c. are straight lines. Hence, by the harmonic properties of a quadrilateral, pr and sq intersect in the pole of TV with respect to any conic through $pqrs$, and therefore the intersection of pr and sq is the pole of the given transversal with regard to the conic of the Question. Similarly, the intersection of sq and vt is the pole of PR, *i. e.* of the transversal, with respect to the same conic. Hence sq, pr, vt meet in the pole of the transversal.

2766. (Proposed by S. WATSON.)—Three points move from a fixed point in three different but given directions, with given uniform but different velocities. Find the locus of the centre of the circle passing through them.

I. *Quaternion Solution by* W. H. LAVERTY, B.A.

Let α, β, γ be the unit-vectors in the directions of motion, and at the end of any time let $kx_1\alpha$, $kx_2\beta$, $kx_3\gamma$ be the vectors to the moving points; so that, ρ being the vector to the centre of the circle, we have

$$(kx_1\alpha - \rho)^2 = (kx_2\beta - \rho)^2 = (kx_3\gamma - \rho)^2,$$

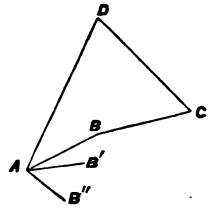
therefore $kx_1^2 + 2x_1 S\alpha\rho = kx_2^2 + 2x_2 S\beta\rho = kx_3^2 + 2x_3 S\gamma\rho$;

hence, eliminating k, and making obvious substitutions, we get

$$S(a\alpha + b\beta + c\gamma)\rho = 0,$$

where $a+b+c=0$; and therefore the locus is a straight line perpendicular to the vector

$$\rho' = x(a\alpha + b\beta + c\gamma).$$



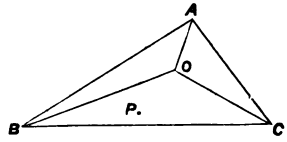
If AB, BC, DC be the directions respectively of α, β, γ; and if, arithmetically, AB + BC = CD; then AD is the direction of ρ'.

II. *Quaternion Solution by* F. D. THOMSON, M.A.; *and* G. A. OGILVIE.

It is easy to see that the locus must be a straight line through the origin, since the triangle formed by the moving points always remains parallel and similar to its initial position; but it may be interesting to give an investigation of the locus by the method of *Quaternions*.

Let O be the point of departure; A, B, C the points at the time t; u, v, w vectors representing the constant velocities of the points. Then, writing α, β, γ for OA, OB, OC, we have

$$\frac{\alpha}{u} = \frac{\beta}{v} = \frac{\gamma}{w} = t.$$



Now, let P be the centre of the circle circumscribing ABC, and let $OP = \rho$.

Then we have to express the facts, first, that P is in the plane ABC, and second, that it is equidistant from ABC.

The first condition is equivalent to the assertion that PA, AB, BC are in one plane, or that $S.PA.AB.BC = 0$, or $S.(\alpha-\rho)(\beta-\alpha)(\gamma-\beta) = 0$, or $S.\rho(V\alpha\beta + V\beta\gamma + V\gamma\alpha) = S\alpha\beta\gamma$, or $S.\rho(Vuv + Vvw + Vwu) = tSuvw$ ... (1).

The second condition gives

$$(\alpha-\rho)^2 = (\beta-\rho)^2 = (\gamma-\rho)^2, \text{ or } \alpha^2 - 2S\rho\alpha = \beta^2 - 2S\rho\beta = \gamma^2 - 2S\rho\gamma.$$

or $$2S\rho(\beta-\alpha) = \beta^2-\alpha^2, \text{ and } 2S\rho(\gamma-\beta) = \gamma^2-\beta^2;$$

or $$2S\rho(v-u) = t(v^2-u^2), \text{ and } 2S\rho(w-v) = t(w^2-v^2) \ldots\ldots (2, 3).$$

Eliminating t,

$$\frac{S.\rho(Vuv + Vvw + Vwu)}{Suvw} = \frac{2S\rho(v-u)}{v^2-u^2} = \frac{2S\rho(w-v)}{w^2-v^2},$$

a system equivalent to two equations, each of the form $S\rho\delta = 0$, and therefore representing two planes through the origin, their intersection being the locus required.

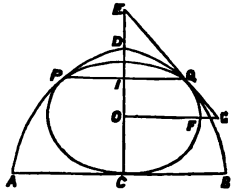
Similarly, it may be shown that the locus of the intersection of perpendiculars of the triangle ABC has for its equations

$$\frac{S.\rho(Vuv + Vvw + Vwu)}{Suvw} = \frac{S\rho(w-v)}{Su(w-v)} = \frac{S\rho(u-w)}{Sv(u-w)}.$$

[The three directions of motion are, in the *first* solution, supposed to be in one plane, but in the *second* solution, anywhere in space.]

2745. (Proposed by the EDITOR.)—Show that the average area of all the ellipses that can be inscribed symmetrically in a portion of a parabola cut off by a double ordinate perpendicular to the axis, in parts of the area of the parabola, is $\frac{23}{160}\pi$ if (1) the centres be equably distributed over all possible positions on the axis, and $\frac{11}{64}\pi$ if (2) the intersections of the chords of contact with the axis be equably distributed on the axis.

I. *Solution by* STEPHEN WATSON.



Let AB be the double ordinate, CD the axis, PQ the chord of contact cutting CD in I, O the centre of the inscribed ellipse, and EG a common tangent at Q, meeting CD produced in E, and the semi-diameter OF, perpendicular to CD produced, in G.

Put CD = a, CB = b, CO = α, OF = β, OI = ax; then we have

$$a^2 = OE.OI = (2ID + OI)OI = 2(a-\alpha)ax - a^2x^2,$$

therefore $\alpha = a\left\{(2x)^{\frac{1}{2}} - x\right\}$, and EI (=2DI) : IQ = EO : OG,

therefore $$\beta^2 = OG.IQ = \frac{EO.IQ^2}{2DI} = \frac{b^2}{2a}EO = \frac{b^2a^2}{2a^2x};$$

therefore $$\beta = b\left\{1 - \left(\frac{x}{2}\right)^{\frac{1}{2}}\right\};$$

hence the area of the ellipse in parts of that of the parabola is

$$\frac{\pi\alpha\beta}{\frac{4}{3}ab} = \frac{3\pi}{4}\left\{(2x)^{\frac{1}{2}} - x\right\}\left\{1 - \left(\frac{x}{2}\right)^{\frac{1}{2}}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

(1). In this case the number of possible positions of O is $\frac{1}{2}a$, and the limits of x are 0 and $\frac{1}{2}$; hence the required average is

$$\frac{2}{a}\int_0^{\frac{1}{2}a} (A)\,da = \frac{3\pi}{2}\int_0^{\frac{1}{2}} \left\{(2x)^{\frac{1}{2}} - x\right\}\left\{1 - \left(\frac{x}{2}\right)^{\frac{1}{2}}\right\}\left\{(2x)^{-\frac{1}{2}} - 1\right\}dx = \frac{23}{160}\pi.$$

(2). In this case the number of positions of I is a; and the limits of x, 0 and $\frac{1}{2}$; hence the required average is

$$\frac{1}{a}\int_0^a (A)\,d(a + ax) = \frac{3\pi}{4}\int_0^{} \left\{(2x)^{\frac{1}{2}} - x\right\}\left\{1 - \left(\frac{x}{2}\right)^{\frac{1}{2}}\right\}(2x)^{-\frac{1}{2}}dx = \frac{11}{64}\pi.$$

II. *Solution by the* PROPOSER.

Let the equation to the parabola be

$$y^2 = 4mx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

then, (α, β) being the semi-axes of an inscribed ellipse, its equation will be

$$\frac{(x - a + \alpha)^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Eliminating y from (1) and (2), and applying the condition for equal roots to the resultant, we find that the ellipse (2) will have double symmetric contact with the given parabola (1) if

$$\beta^2 + 2ma = b\beta, \quad \text{or} \quad a = \frac{b\beta - \beta^2}{2m} \quad \ldots\ldots\ldots\ldots\ldots (3).$$

Otherwise: The equation of a conic which has double contact with the parabola (1), along the chord PQ or $x - z = 0$ where $z = \text{DI}$, is

$$y^2 - 4mx = k(x-z)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4);$$

and determining k so that (4) may touch the double ordinate AB, we find

$$k = -\frac{4ma}{(a-z)^2} = -\frac{b^2}{(a-z)^2};$$

hence the equation of an ellipse inscribed in the parabola becomes

$$\frac{a}{m(a+z)^2}\, y^2 + \frac{4a^2}{(a^2-z^2)^2}\left(x - \frac{a^2+z^2}{2a}\right)^2 = 1 \quad \ldots\ldots\ldots (5);$$

and its semi-axes are $\quad \alpha = \dfrac{a^2-z^2}{2a}, \quad \beta = \dfrac{2m(a+z)}{b} \quad \ldots\ldots\ldots\ldots\ldots (6).$

From (3) and (6) we see that the area of the ellipse, in parts of the parabola, is

$$\frac{3\pi}{2b^4}(b\beta^2 - \beta^3), \quad \text{or} \quad \frac{3\pi}{16a^3}(a+z)(a^2-z^2) \ldots\ldots\ldots (7, 8).$$

In (1), a varies uniformly from 0 to $\frac{1}{2}a$; hence, by (7), the average area of the inscribed ellipses is

$$\frac{3\pi}{ab^3}\int_0^{\frac{1}{2}a} (b\beta - \beta^3)\, d\beta = \frac{6\pi}{b^5}\int_b^{\frac{1}{2}b} (b\beta - \beta^3)(b - 2\beta)\, d\beta$$

$$= \frac{6\pi}{b^5}\int_{\frac{1}{2}b}^{b} (-2\beta^4 + 3b\beta^3 - b^2\beta^2)\, d\beta = \frac{23}{160}\pi.$$

In (2), z varies uniformly from 0 to a; hence, by (8), the average area of the inscribed ellipses is

$$\frac{3\pi}{16a^4}\int_0^a (a^3 + a^2 z - az^2 - z^3)\, dz = \frac{11}{64}\pi.$$

From (7) or (8) we find that the inscribed ellipse is a maximum when $z = \frac{1}{3}a$, $\alpha = \frac{4}{9}a$, $\beta = \frac{2}{3}b$; its area being $\frac{2}{9}\pi$.

2640. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—1. If a conic of given eccentricity e circumscribe a triangle ABC, the locus of its centre is

$$\frac{(yz\sin^2 A + zx\sin^2 B + xy\sin^2 C)^2}{xyz\,(x+y+z)\sin^2 A \sin^2 B \sin^2 C} = \frac{(e^2-2)^2}{1-e^2};$$

and (2) if it be inscribed in the triangle, the locus of the centre is

$$\frac{(x^2 \sin^2 A + \ldots + 2yz \sin B \sin C \cos A + \ldots)^2}{xyz\,(x+y+z) \sin^2 A \sin^2 B \sin^2 C} = \frac{4\,(e^2-2)^2}{1-e^2};$$

the triangle of reference having its angular points at the middle points of the sides of the triangle ABC.

3. Hence show that the locus of points with which as centre two similar conics can be drawn, one inscribed and the other circumscribed to the triangle, is two circles, one the circumscribed circle of the triangle, and the other that which has the centre of gravity and the centre of perpendiculars of the triangle as the ends of a diameter.

I. *Solution by* W. S. M'CAY, B.A., *and the* PROPOSER.

1. If Σ and Ω be the tangential equations of a conic and the circular points at infinity; then the values of k^2, for which $\Sigma + k^2\Omega = 0$ represents points, are proportional to the squares of the axes of the conic, and are given by the discriminating equation

$$\Delta^2 + \Delta\Theta' k^2 + \Theta k^4 = 0,$$

for this is evidently so in the case $a^2\lambda^2 + b^2\mu^2 - \nu^2 + k^2(\lambda^2+\mu^2) = 0$.

Hence, easily, if ϕ be the angle between the asymptotes

$$\frac{(e^2-2)^2}{e^2-1} = 4\cot^2\phi = \frac{\Theta'^2}{-\Theta},$$

Θ being positive for an ellipse and negative for an hyperbola.

Now for the circumscribing conic $2(fyz+gxz+hxy) = 0$,

$$-\Theta = f^2 \sin^2 A + \ldots - 2gh \sin B \sin C \ldots,$$

$$-\Theta' = 2(f \cos A + g \cos B + h \cos C).$$

But $$f : g : h = \frac{X}{\sin A}(-X+Y+Z) : (\ ) : (\ ),$$

in terms of areal coordinates of centre,

or $$f : g : h = \frac{y+z}{\sin A}x \ ; \ \frac{x+z}{\sin B}y \ : \ \frac{x+y}{\sin C}z,$$

the coordinates being referred to the triangle whose vertices are the middle points of the sides of the original triangle. Hence we obtain at once the first equation given in the question.

2. Again, for the inscribed conic $(lx)^{\frac{1}{2}} + (my)^{\frac{1}{2}} + (nz)^{\frac{1}{2}} = 0$,

$$\Theta = 2lmn(l \sin B \sin C + m \sin A \sin C + n \sin A \sin B),$$

$$\Theta' = l^2 + m^2 + n^2 + 2mn \cos A + \ldots,$$

and $$l : m : n = \sin A(-X+Y+Z) : (\ ) : (\ ) = x \sin A : y \sin B : z \sin C.$$

Hence we obtain the second equation given in the question.

3. Eliminating e from the equations in (1) and (2), we see that if with the same point (x, y, z) as centre there can be drawn two similar conics, one inscribed in the triangle ABC and the other circumscribed about it, the point (x, y, z) must lie on one of the two circles

$$x^2 \sin^2 A + \ldots + 2yz \sin B \sin C \cos A + \ldots = \pm 2(yz \sin^2 A + \ldots).$$

If the lower sign be taken, this is the circumscribed circle of the triangle ABC; and if the upper, it is the circle in which the centre of perpendiculars and the centre of gravity are ends of a diameter. This latter circle is therefore one of the system coaxal with the circumscribed and nine-point circles.

II. *Solutions by the* PROPOSER.

1. The equation of a conic circumscribing the triangle ABC, referred to AB, AC as axes of Cartesian coordinates, is of the form

$$\lambda X(X-c)+\mu XY+\nu Y(Y-b)=0;$$

and if e be the eccentricity,

$$\frac{(e^2-2)^2}{1-e^2}=\frac{4(\lambda+\nu-\mu\cos A)^2}{\sin^2 A\,(4\nu\lambda-\mu^2)};$$

also for its centre $\quad \lambda(2X-c)+\mu Y=\nu(2Y-b)+\mu X=0;$

but if we transfer to areal coordinates measured on the triangle abc (the middle points of the sides of ABC),

$$2\frac{X}{c}=z+x, \quad 2\frac{Y}{b}=y+x;$$

and therefore for the centre

$$\frac{\lambda c^2}{(x+y)z}=\frac{\nu b^2}{y(x+z)}=\frac{\mu bc}{2yz},$$

and the locus of the centre, when the eccentricity e is given, is

$$\frac{(e^2-2)^2}{1-e^2}=\frac{\left\{z(x+y)\sin^2 B+y(x+z)\sin^2 C-2yz\sin B\sin C\cos A\right\}^2}{\sin^2 A\sin^2 B\sin^2 C\left\{yz(x+y)(x+z)-y^2z^2\right\}},$$

which easily reduces to the form given in the question.

2. The equation of an inscribed conic is

$$\frac{X}{h}+\frac{Y}{k}-1=2\lambda\left(\frac{XY}{hk}\right)^{\frac{1}{2}},$$

with the condition $\quad \left(\frac{1}{h}-\frac{1}{c}\right)\left(\frac{1}{k}-\frac{1}{b}\right)=\frac{\lambda^2}{hk};$

and for the centre $\quad \frac{X}{h}=\frac{Y}{k}=\frac{1}{2(1-\lambda^2)};$

and combining these, we find

$$\frac{h}{X}=\frac{k}{Y}=2(1-\lambda^2)=\frac{bc}{XY}\left(\frac{X}{c}+\frac{Y}{b}-\frac{1}{2}\right).$$

Transforming as before to the triangle abc, we find

$$\lambda^2=\frac{yz}{(x+y)(x+z)}, \quad \frac{h}{c}(x+y)=\frac{k}{b}(x+z)=x(x+y+z).$$

But, for the eccentricity e, we have the equation

$$\frac{(e^2-2)^2}{1-e^2}=\frac{\left\{\frac{1}{h^2}+\frac{1}{k^2}-\frac{2}{hk}(1-2\lambda^2)\cos A\right\}^2}{\sin^2 A\left\{\frac{1}{h^2k^2}-\frac{(1-2\lambda^2)^2}{h^2k^2}\right\}}$$

$$\equiv\frac{\left\{(x+y)^2\sin^2 B+(x+z)^2\sin^2 C-2(x+y)(x+z)\sin B\sin C\cos A+4\sin B\sin C\cos A\,yz\right\}^2}{4\sin^2 A\sin^2 B\sin^2 C\,.\,yz\,.\,x(x+y+z)}$$

$$=\frac{(x^2\sin^2 A+\ldots.+2yz\sin B\sin C\cos A+\ldots.)^2}{4xyz(x+y+z)\sin^2 A\sin^2 B\sin^2 C}.$$

Another solution may be obtained in the following manner:—

1. If the triangle of reference be the triangle having its angles at the middle points of the sides of the triangle ABC, the equation of a conic circumscribing ABC is

$$l(x+y)(x+z)+m(y+z)(y+x)+n(z+x)(z+y)=0,$$

and the equation of the straight lines through A parallel to its asymptotes is

$$lyz-mz(y+z)-ny(y+z)=0;$$

whence, if ϕ be the angle between the asymptotes,

$$\tan^2\phi=\frac{\sin^2 A\sin^2 B\sin^2 C\{(m+n-l)^2-4mn\}}{\{n\sin^2 B-(m+n-l)\sin B\sin C\cos A+m\sin^2 C\}^2}.$$

Now, if e be the eccentricity and (X, Y, Z) the centre,

$$\frac{(e^2-2)^2}{1-e^2}=-4\cot^2\phi, \quad \text{and} \quad \frac{l}{X(Y+Z)}=\frac{m}{Y(Z+X)}=\frac{n}{Z(X+Y)};$$

whence, if e be given, the locus of the centre is

$$\frac{1-e^2}{(e^2-2)^2}=\frac{\sin^2 A\sin^2 B\sin^2 C\{YZ(Z+X)(X+Y)-Y^2Z^2\}}{\{Z(X+Y)\sin^2 B-2YZ\sin B\sin C\cos A+Y(Z+X)\sin^2 C\}^2}$$
$$\equiv\frac{XYZ(X+Y+Z)\sin^2 A\sin^2 B\sin^2 C}{\{YZ\sin^2 A+ZX\sin^2 B+XY\sin^2 C\}^2}.$$

2. With a similar notation, the equation of an inscribed conic being

$$\{l(y+z)\}^{\frac{1}{2}}+\{m(z+x)\}^{\frac{1}{2}}+\{n(x+y)\}^{\frac{1}{2}}=0,$$

$$\tan^2\phi=-\frac{16lmn(l+m+n)\sin^2 A\sin^2 B\sin^2 C}{\{l^2\sin^2 A+m^2\sin^2 B+n^2\sin^2 C+2mn\sin B\sin C\cos A+\ldots+\ldots\}^2},$$

and $\frac{X}{l}=\frac{Y}{m}=\frac{Z}{n}$, whence the locus of the centre of an inscribed conic of given eccentricity e is

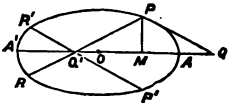
$$\frac{(e^2-2)^2}{1-e^2}=\frac{(X^2\sin^2 A+\ldots+2YZ\sin B\sin C\cos A+\ldots)^2}{4XYZ(X+Y+Z)\sin^2 A\sin^2 B\sin^2 C}.$$

2781. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—Prove that the maximum value of the common chord of an ellipse and its circle of curvature, a and b being the semiaxes of the ellipse, is

$$\frac{4}{3\sqrt{3}(a^2-b^2)}\left\{2(a^4-a^2b^2+b^4)^{\frac{3}{2}}-(a^2+b^2)(2a^2-b^2)(2b^2-a^2)\right\}^{\frac{1}{2}}.$$

I. *Solution by* MATTHEW COLLINS, B.A.

Let $AA'=2a$ be the major axis, O the centre, and $PQ'=t=$ the tangent PQ; then PQ'R is well known to be a chord of the circle osculating the ellipse at P. Let $OM=x$, the perpendicular $MP=y$; then we have



$$OQ=\frac{a^2}{x},\quad MQ=MQ'=\frac{a^2}{x}-x,\quad t^2=\left(\frac{a^2}{x}-x\right)^2+y^2=\left(\frac{a^2}{x}-x\right)^2+b^2-\frac{b^2x^2}{a^2}.$$

Again, drawing P'Q'R' parallel to PQ, and putting Q'R = z, we have

$$\frac{QA.QA'}{QP^2}=\frac{Q'A.Q'A'}{Q'P'.Q'R'}=\frac{Q'A.Q'A'}{Q'P.Q'R'},\quad i.e.\ \frac{\frac{x^4}{a^2}-a^2}{t^2}=\frac{a^2-\left(\frac{a^2}{x}-2x\right)^2}{tz},$$

therefore
$$\frac{z}{t}=\frac{a^2-\left(\frac{a^2}{x}-2x\right)^2}{\frac{a^4}{x^2}-a^2}=\frac{(a^2-x^2)(4x^2-a^2)}{a^2(a^2-x^2)}=\frac{4x^2-a^2}{a^2};$$

therefore
$$PR=t+z=t\left(1+\frac{z}{t}\right)=t\left(\frac{4x^2}{a^2}\right).$$

Now, when this is a maximum,

$$x^4t^2=x^4\left\{\left(\frac{a^2}{x}-x\right)^2+b^2-\frac{b^2x^2}{a^2}\right\}=\frac{1}{a^2}\left\{(a^2-b^2)x^6-a^2(2a^2-b^2)x^4+a^6x^2\right\}$$

must also be a maximum; its differential coefficient equated to zero gives

$$3(a^2-b^2)x^4-2a^2x^2(2a^2-b^2)+a^6=0,\quad \text{whence } \frac{a^4}{x^2}=2a^2-b^2\pm R=OQ^2,$$

where $R=(a^4-a^2b^2+b^4)^{\frac{1}{2}}$ is plainly $>a^2-\frac{1}{2}b^2$; and therefore, as $OQ^2>a^2$, the *proper* value of OQ^2 is $2a^2-b^2+R$; hence

$$t^2=\frac{a^4}{x^2}-2a^2+b^2+\frac{x^2}{a^2}(a^2-b^2)=R+\frac{a^2(a^2-b^2)}{2a^2-b^2+R}$$
$$=R+\frac{2a^2-b^2-R}{3}=\frac{2R+2a^2-b^2}{3};$$

therefore
$$PR^2=\frac{16}{a^4}t^2x^4=\frac{16a^4t^2}{\left(\frac{a^4}{x^2}\right)^2}=\frac{16a^4}{3}\cdot\frac{2R+2a^2-b^2}{(R+2a^2-b^2)^3}$$
$$=\frac{16a^4}{3}\cdot\frac{(2R+2a^2-b^2)(2a^2+b^2-R)^2}{\left\{3a^2(a^2-b^2)\right\}^2}$$
$$=\frac{16}{27(a^2-b^2)^2}\left\{2R^3-3R^2(2a^2-b^2)+(2a^2-b^2)^3\right\}$$
$$=\frac{16}{27(a^2-b^2)^2}\left\{2R^3-(2a^2-b^2)\left[3R^2-(2a^2-b^2)^2\right]\right.$$
$$=\frac{16}{27(a^2-b^2)^2}\left\{2(a^4-a^2b^2+b^4)^{\frac{3}{2}}-(a^2+b^2)(2a^2-b^2)(2b^2-a^2)\right\},$$

the required expression for the square of the maximum common chord.

II. *Solution by the* Proposer.

The circle of curvature of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, at the point whose eccentric angle is θ, meets the ellipse again in the point $(a\cos 3\theta,\ -b\sin 3\theta)$, and the square on the common chord is therefore

$c^2 = \frac{1}{4}(x^2+y^2)^2 - \frac{1}{2}(a^2x^2\cos^2\theta + b^2y^2\sin^2\theta) = \frac{1}{2}(1-z^2)(a^2+b^2-c^2z)$, if $z = \cos 2\theta$.

For a maximum or minimum value,

either $\frac{dz}{d\theta} = 0$, or $3c^2z^2 - 2(a^2+b^2)z - c^2 = 0$,

of which the former gives $z=0$; and since c^2 is essentially positive, this is the minimum value. The maximum value is then given by that root of the latter equation which is numerically less than 1, or by

$$z = \frac{a^2+b^2-2\sqrt{(a^4-a^2b^2+b^4)}}{3(a^2-b^2)} \equiv \frac{a^2+b^2-2R}{3c^2}$$

For this value we have

$$\begin{aligned}
c^2 &= \frac{2z}{3}\{2(a^2+b^2)z - c^2\} - \frac{2(a^2+b^2)}{3c^2}\{2(a^2+b^2)z + c^2\} - 2c^2z + 2(a^2+b^2) \\
&= \frac{4(a^2+b^2)}{9c^2}\{2(a^2+b^2)z - c^2\} - \frac{4(a^2+b^2)}{3c^2}z - \frac{4c^2z}{3} + \frac{4(a^2+b^2)}{3} \\
&= -\frac{4}{9c^2}z\{(a^2+b^2)^2 + 3(a^2-b^2)^2\} + \frac{16}{9}(a^2+b^2) \\
&= -\frac{16}{27c^4}(a^4-a^2b^2+b^4)(a^2+b^2-2R) + \frac{16}{9}(a^2+b^2) \\
&= \frac{16}{27c^4}\left\{2R^3 - (a^2+b^2)\left[(a^4-a^2b^2+b^4) - 3(a^2-b^2)^2\right]\right\} \\
&= \frac{16}{27(a^2-b^2)^2}\{2(a^4-a^2b^2+b^4)^{\frac{3}{2}} - (a^2+b^2)(2a^2-b^2)(2b^2-a^2)\}.
\end{aligned}$$

The longest common chord has therefore the value given in the question.

2345. (Proposed by H. R. Greer, B.A.)—Determine in trilinear coordinates the foot of the perpendicular from the point (x', y', z') on the line $lx + my + nz = 0$; also the reflexion of the same point with regard to the same line.

Solution by James Dale.

If the line $\quad Lx + My + Nz = 0 \ldots\ldots\ldots\ldots\ldots (1)$

be perpendicular to $\quad lx + my + nz = 0 \ldots\ldots\ldots\ldots\ldots (2)$,

and pass through (x', y', z'), we have the conditions

$$Lx' + My' + Nz' = 0, \quad Ll' + Mm' + Nn' = 0$$

(where $l' = l - n\cos B - m\cos C$, with corresponding values for m', n');

therefore $$\frac{L}{\begin{vmatrix} m' & n' \\ y' & z' \end{vmatrix}} = \frac{M}{\begin{vmatrix} n' & l' \\ z' & x' \end{vmatrix}} = \frac{N}{\begin{vmatrix} l' & m' \\ x' & y' \end{vmatrix}}.$$

The intersection of (1) and (2), that is, the foot of the perpendicular from the given point on the given line, is

$$\frac{\xi}{\begin{vmatrix} m & n \\ M & N \end{vmatrix}} = \frac{\eta}{\begin{vmatrix} n & l \\ N & L \end{vmatrix}} = \frac{\zeta}{\begin{vmatrix} l & m \\ L & M \end{vmatrix}};$$

and substituting for L, M, N, we get

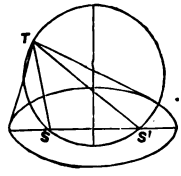
$$\frac{\xi}{\begin{vmatrix} m & n \\ \begin{vmatrix} n' & l' \\ z' & x' \end{vmatrix} & \begin{vmatrix} l' & m' \\ x' & y' \end{vmatrix} \end{vmatrix}} = \frac{\eta}{\begin{vmatrix} n & l \\ \begin{vmatrix} l' & m' \\ x' & y' \end{vmatrix} & \begin{vmatrix} m' & n' \\ y' & z' \end{vmatrix} \end{vmatrix}} = \frac{\zeta}{\begin{vmatrix} l & m \\ \begin{vmatrix} m' & n' \\ y' & z' \end{vmatrix} & \begin{vmatrix} n' & l' \\ z' & x' \end{vmatrix} \end{vmatrix}}$$

$$= \frac{2\Delta}{\begin{vmatrix} a & b & c \\ l & m & n \\ \begin{vmatrix} m' & n' \\ y' & z' \end{vmatrix} & \begin{vmatrix} n' & l' \\ z' & x' \end{vmatrix} & \begin{vmatrix} l' & m' \\ x' & y' \end{vmatrix} \end{vmatrix}}$$

The coordinates of the point of reflexion (x_1, y_1, z_1) may now be found from the equations $2\xi = x' + x_1$, $2\eta = y' + y_1$, $2\zeta = z' + z_1$.

2301. (Proposed by W. K. CLIFFORD.)—A circle is drawn so that its radical axis with respect to the focus S of a parabola is a tangent to the parabola; if a tangent to the circle cut the parabola in A, B, and if SC, bisecting the angle ASB, cut AB in C, the locus of C is a straight line.

Solution by the REV. J. WOLSTENHOLME, M.A.



Through the foci S, S′ of a conic draw a circle, and from any point T on this circle draw two tangents; the straight lines bisecting the angles between these tangents will bisect the angles between TS and TS′, and will therefore pass through the fixed points where the circle meets the minor axis. Reciprocate the system on S; then to the circle corresponds a parabola having S for its focus, and to the ellipse a circle having for radical axis to it and the point-circle S a tangent to the parabola (the reciprocal of S′); to the point T corresponds a tangent to the parabola; to the tangents the two points A, B; and to the straight lines bisecting the angle, points on AB where the straight lines bisecting ASB meet it. Whence the theorem.

2456. (Proposed by Dr. SHAW.)—Show that if an ellipse pass through the centre of a hyperbola, and have its foci on the hyperbola's asymptotes; (*a*) the hyperbola passes through the centre of the ellipse, (*b*) the axes of each curve are respectively tangent and normal to each other, and (*c*) the two axes which are also normals are equal to each other.

Solution by W. H. LAVERTY, B.A.

(a) Let the equation to the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; therefore the equations to the asymptotes of the hyperbola are

$$xy'-y(x'-c)-cy'=0, \text{ and } xy'-y(x'+c)+cy'=0,$$

where the centre (x', y') of the hyperbola is on the ellipse. Hence the equation to the hyperbola is

$$x^2y'^2+y^2(x'^2-c^2)-2xy\,x'y'+2c^2yy'+k^4=0,$$

where k^4 is a constant. The curve then will not pass through the centre of the ellipse unless $k=0$, when the equation becomes

$$x^2y'^2+y^2(x'^2-c^2)-2xy\,x'y'+2c^2yy'=0 \ldots\ldots\ldots\ldots\ldots (1).$$

(b.) If, in (1), we put $y=0$, we have $x^2=0$; therefore the major axis of the ellipse is a tangent to the hyperbola at the origin, and the minor axis is a normal; also the axes of (1), being the bisectors of the focal distances, are tangent and normal to the ellipse.

(c.) Again, transform the equation (1) so that (x', y') may be the origin, and the tangent and normal to the ellipse the coordinate axes; the transformed equation will be $\frac{x^2}{b^2}-\frac{y^2}{c^2-c^2x'^2}=1.$

Therefore the axis of the hyperbola, which is a normal to the ellipse, is equal to b, and is equal to the minor axis of the ellipse, which, by (b), is the normal to the hyperbola.

2653. (Proposed by J. GRIFFITHS, M.A.)—1. If the centre of one of the four circles which touch the sides of a triangle self-conjugate with respect to a parabola lies on the directrix; show that its circumference passes through the focus.

2. If the point of intersection of the three perpendiculars of a triangle inscribed in a parabola lies on the directrix; show that the self-conjugate circle of this triangle passes through the focus of the curve.

Solution by the REV. J. WOLSTENHOLME, M.A.

1. If we take the triangle as that of reference, the equation of the parabola is $lx^2+my^2+nz^2=0$, with the condition $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}=0.$

The equation of the directrix is $x\left(\frac{l^2}{n}+\frac{c^2}{m}\right)+\ldots.+\ldots.=0;$

which will pass through the centre of the inscribed circle, if

$$a\left(\frac{b^2}{n}+\frac{c^2}{m}\right)+\ldots.+\ldots.=0, \text{ or } \frac{b+c}{al}+\ldots.+\ldots.=0.$$

Hence $l\left(\frac{c+a}{b}-\frac{a+b}{c}\right)=m\left(\frac{a+b}{c}-\frac{b+c}{a}\right)=n\left(\frac{c+a}{b}-\frac{a+b}{c}\right),$

or $$la(b-c)=mb(c-a)=nc(a-b).$$

But the focus is given by the equations

$$\frac{x}{mb^2+nc^2} = \ldots\ldots, \quad \text{or} \quad \frac{x}{\frac{b}{c-a}+\frac{c}{a-b}} = \ldots\ldots = \ldots\ldots$$

or $$\frac{x}{(b-c)^2(b+c-a)} = \frac{y}{(c-a)^2(c+a-b)} = \frac{z}{(a-b)^2(a+b-c)};$$

and this point manifestly lies on the inscribed circle $(x\cot\frac{1}{2}A)^{\frac{1}{2}}+\ldots+\ldots=0$. This point is of course the point of contact of the inscribed, or escribed, circle with the nine-point circle. It may be noticed that the tangent to this circle at the focus of the parabola is the polar with respect to the parabola of the centre of the circle, its equation being $\frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b}=0$.

There is no difference in the work when one of the other circles is taken, beyond changing the signs of one of the three, a, b, c.

2. This is only the reciprocal of the above proposition with respect to the focus of the parabola.

2718. (Proposed by Professor CAYLEY.)—Find *in plano* the locus of a point P, such that from it two given points A, C, and two given points B, D, subtend equal angles.

2757. (Proposed by Professor CAYLEY.)—If

$$x_0^2+y_0^2=1, \quad x_1^2+y_1^2=1, \quad \text{and} \quad \begin{vmatrix} x, & y, & 1 \\ x_0, & y_0, & 1 \\ x_1, & y_1, & 1 \end{vmatrix} = \mathrm{L};$$

show that each of the equations

$$\frac{a^2(x-x_0)^2+b^2(y-y_0)^2}{(xx_0+yy_0-1)^2} = \frac{a^2(x-x_1)^2+b^2(y-y_1)^2}{(xx_1+yy_1-1)^2} \quad \ldots\ldots\ldots(1),$$

$$\frac{a^2(x-x_0)^2+b^2(y-y_0)^2}{(xy_0-x_0y)^2-(x-x_0)^2-(y-y_0)^2} = \frac{a^2(x-x_1)^2+b^2(y-y_1)^2}{(xy_1-x_1y)^2-(x-x_1)^2-(y-y_1)^2} \quad \ldots(2),$$

represents the right line $\mathrm{L}=0$ and a cubic curve.

1819. (Proposed by C. TAYLOR, M.A.)—From two fixed points on a given conic pairs of tangents are drawn to a variable confocal conic, and with the fixed points as foci a conic is described passing through any one of the four points of intersection. Show that its tangent or normal at that point passes through a fixed point.

Solution of the above Problems by PROFESSOR CAYLEY.

1. It is easy to see that drawing through the points A, C a circle, and through B, D a circle, such that the radii of the two circles are proportional to the lengths AC, BD, then that the required locus is that of the intersections of the two variable circles.

Take AC = 2l, MO perpendicular to it at its middle point M, and = p; a, b the coordinates of M, and λ the inclination of p to the axis of x; then

coordinates of O are $a+p\cos\lambda$, $b+p\sin\lambda$,
coordinates of A, C are $a\pm l\sin\lambda$, $b\mp l\cos\lambda$,

and hence the equation of a circle, centre O and passing through A, C, is

$$(x-a-p\cos\lambda)^2+(y-b-p\sin\lambda)^2=l^2+p^2;$$

or, what is the same thing,

$$(x-a)^2+(y-b)^2-l^2=2p[(x-a)\cos\lambda+(y-b)\sin\lambda].$$

If $2m$, q, c, d, μ refer in like manner to the points B, D, then the equation of a circle centre say Q, and passing through B, D, is

$$(x-c)^2+(y-d)^2-m^2=2q[(x-c)\cos\mu+(y-d)\sin\mu].$$

And the condition as to the radii is $l^2+p^2 : m^2+q^2=l^2 : m^2$, that is, $p^2 : q^2= l^2 : m^2$, or $p : q = \pm l : m$. And we thus have for the equation of the required locus

$$\frac{(x-a)^2+(y-b)^2-l^2}{(x-a)\cos\lambda+(y-b)\sin\lambda}=\pm\frac{l}{m}\frac{(x-c)^2+(y-d)^2-m^2}{(x-c)\cos\mu+(y-d)\sin\mu},$$

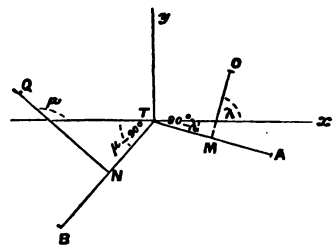
viz., the locus is composed of two cubics, which are at once seen to be *circular* cubics. One of these will however belong (at least for some positions of the four points) to the case of the subtended angles being *equal*, the other to that of the subtended angles being *supplementary*; and we may say that the required locus is a circular cubic.

2. If two of the points coincide, suppose C, D at T; then, taking T as the origin, we may write

$$a = l\sin\lambda,\quad b=-l\cos\lambda,$$
$$c=-m\sin\mu,\quad d=m\cos\mu,$$

and the equation becomes

$$\frac{x^2+y^2=2l(x\sin\lambda-y\cos\lambda)}{x\cos\lambda+y\sin\lambda}$$
$$=\pm\frac{l}{m}\frac{x^2+y^2+2m(x\sin\mu-y\cos\mu)}{x\cos\mu+y\sin\mu}$$



viz., this is

$$(x^2+y^2)[m(x\cos\mu+y\sin\mu)\mp l(x\cos\lambda+y\sin\lambda)]$$
$$-2lm\{(x\sin\lambda-y\cos\lambda)(x\cos\mu+y\sin\mu)$$
$$\pm(x\sin\mu-y\cos\mu)(x\cos\lambda+y\sin\lambda)\}=0.$$

Taking the lower signs, the term in $\{\}$ is $(x^2+y^2)\sin(\lambda-\mu)$, and the equation is

$$(x^2+y^2)\{m(x\cos\mu+y\sin\mu)+l(x\cos\lambda+y\sin\lambda)-2lm\sin(\lambda-\mu)\}=0,$$

viz., this is $x^2+y^2=0$, and a line which is readily seen to be the line AB; and in fact from any point whatever of this line the points A, T and the points B, T subtend *supplementary* angles.

Taking the upper signs, the equation is

$$(x^2+y^2)[m(x\cos\mu+y\sin\mu)-l(x\cos\lambda+y\sin\lambda)]$$
$$-2lm\{(x^2-y^2)\sin(\lambda+\mu)-xy\cos(\lambda+\mu)\}=0,$$

which is the locus for *equal* angles, a circular cubic as in the case of the four distinct points.

3. The Question is connected with Question 1819, which is given above. In fact, taking A, B for the fixed points on the given conic, and P for the intersection of any two of the tangents, if in the conic (foci A, B) which passes

through P, the tangent or normal at P passes through a fixed point T, then it is clear that at P the points A, T and B, T subtend equal angles, and consequently the locus of P should be a circular cubic as above. The theorem will therefore be proved if it be shown that the locus of P considered as the intersection of tangents from A, B to the variable confocal conic is in fact the foregoing circular cubic. I remark that the fixed point T is in fact the intersection of the tangents AT, BT to the given conic at the points A, B respectively.

4. Consider the points A, B, (which we may in the first instance take to be arbitrary points, but we shall afterwards suppose them to be situate on the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,) and from each of them draw a pair of tangents to the confocal conic $\frac{x^2}{a^2+h}+\frac{y^2}{b^2+h}=1$. Take (x_0, y_0) for the coordinates of A, and (x_1, y_1) for those of B; then the equation of the pair of tangents from A is

$$\left(\frac{x_0^2}{a^2+h}+\frac{y_0^2}{b^2+h}-1\right)\left(\frac{x^2}{a^2+h}+\frac{y^2}{b^2+h}-1\right)-\left(\frac{xx_0}{a^2+h}+\frac{yy_0}{b^2+h}-1\right)^2=0,$$

or, what is the same thing,

$$\frac{(xy_0-x_0y)^2}{(a^2+h)(b^2+h)}-\frac{(x-x_0)^2}{a^2+h}-\frac{(y-y_0)^2}{b^2+h}=0,$$

that is, $$(xy_0-x_0y)^2-(b^2+h)(x-x_0)^2-(a^2+h)(y-y_0)^2=0,$$

or as this may also be written

$$(xy_0-x_0y)^2-b^2(x-x_0)^2-a^2(y-y_0)^2=h[(x-x_0)^2+(y-y_0)^2];$$

and similarly for the tangents from B we have

$$(xy_1-x_1y)^2-b^2(x-x_1)^2-a^2(y-y_1)^2=h[(x-x_1)^2+(y-y_1)^2];$$

in which equations the points (x_0, y_0), (x_1, y_1) are in fact any two points whatever.

5. Eliminating h, we have as the locus of the intersection of the tangents

$$\frac{(xy_0-x_0y)^2-b^2(x-x_0)^2-a^2(y-y_0)^2}{(x-x_0)^2+(y-y_0)^2}=\frac{(xy_1-x_1y)^2-b^2(x-x_1)^2-a^2(y-y_1)^2}{(x-x_1)^2+(y-y_1)^2},$$

which is apparently a quartic curve; but it is obvious *à priori* that the locus includes as part of itself the line AB which joins the two given points. In fact, there is in the series of confocal conics one conic which touches the line in question, and since for this conic one of the tangents from A and also one of the tangents from B is the line AB, we see that every point of the line AB belongs to the required locus. The locus is thus made up of the line in question and of the cubic curve.

6. To effect the reduction it will be convenient to write ax, by in the place of x, y, $(ax_0, by_0, ax_1, by_1$ in place of $x_0, y_0, x_1, y_1,)$ and thus consider the equation under the form

$$\frac{a^2(x-x_0)^2+b^2(y-y_0)^2}{(xy_0-x_0y)^2-(x-x_0)^2-(y-y_0)^2}=\frac{a^2(x-x_1)^2+b^2(y-y_1)^2}{(xy_1-x_1y)^2-(x-x_1)^2-(y-y_1)^2}$$

It is to be shown that this equation represents the line $L=0$, and a cubic curve.

Writing for a moment $x_0 = x + \xi_0$, $y_0 = y + \eta_0$, and $x_1 = x + \xi$, $y_1 = y + \eta_1$, the equation becomes

$$\frac{a^2\xi_0^2 + b^2\eta_0^2}{(x\eta_0 - y\xi_0)^2 - \xi_0^2 - \eta_0^2} = \frac{a^2\xi_1^2 + b^2\eta_1^2}{(x\eta_1 - y\xi_1)^2 - \xi_1^2 - \eta_1^2},$$

and hence, multiplying out, the equation is at once seen to contain the factor $\xi_0\eta_1 - \xi_1\eta_0$ (which is in fact the determinant just mentioned), and when divested of this factor the equation is

$$a^2\,[(x^2-1)\,(\xi_0\eta + \xi_1\eta_0) - 2xy\xi_0\xi_1]$$
$$-b^2\,[(y^2-1)\,(\xi_0\eta_1 + \xi_1\eta_0) - 2xy\eta_0\eta_1].$$

Writing herein for ξ_0, η_0, ξ_1, η_1 their values, and consequently

$$\xi_0\xi_1 = x^2 - x\,(x_0 + x_1) + x_0x_1,$$
$$\eta_0\eta_1 = y^2 - y\,(y_0 + y_1) + y_0y_1,$$
$$\xi_0\eta_1 + \xi_1\eta_0 = 2xy - x\,(y_0 + y_1) - y\,(x_0 + x_1) + x_0y_1 + x_1y_0,$$

and arranging the terms, the equation is found to be

$$(a^2x^2 + b^2y^2)\,[-x\,(y_1 + y_0) - y\,(x_1 + x_0)] + (a^2x^2 + b^2y^2)\,(x_0y_1 + x_1y_0)$$
$$-2xy\,[a^2\,(1 + x_0x_1) - b^2\,(1 + y_0y_1)]$$
$$+ (a^2 - b^2)\,[x\,(y_1 + y_0) + y\,(x_1 + x_0) - (x_0y_1 + x_1y_0)] = 0,$$

which is the required cubic curve.

7. Restoring the original coordinates, or writing $\frac{x}{a}$, $\frac{y}{b}$, $\frac{x_0}{a}$, &c. in place of x, y, x_0, &c., we have

$$(x^2 + y^2)\,[-x\,(y_1 + y_0) + y\,(x_1 + x_0)]$$
$$+ (x^2 - y^2)\,(x_0y_1 + x_1y_0) - 2xy\,(a^2 - b^2 + x_0x_1 - y_0y_1)$$
$$+ (a^2 - b^2)\,[x\,(y_1 + y_0) + y\,(x_1 + x_0) - (x_0y_1 + x_1y_0)] = 0,$$

which is a circular cubic the locus of the intersections of the tangents from the arbitrary points (x_0, y_0), (x_1, y_1) to the series of confocal conics $\frac{x^2}{a^2 + h} + \frac{y^2}{b^2 + h} = 1$; the origin of the coordinates is at the centre of the conics.

8. Supposing that the points (x_0, y_0), (x_1, y_1) are on the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and that we have consequently $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$, $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$, the equations of the tangents at these points respectively are

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1, \quad \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1;$$

and hence, writing for shortness $\alpha = y_0 - y_1$, $\beta = x_1 - x_0$, $\gamma = x_0y_1 - x_1y_0$, we find $x = -\frac{a^2\alpha}{\gamma}$, $y = -\frac{b^2\beta}{\gamma}$ as the coordinates of the point of intersection T, of the two tangents; and in order to transform to this point as origin, we must in place of x, y write $x - \frac{a^2\alpha}{\gamma}$, $y - \frac{b^2\beta}{\gamma}$ respectively. Or what is more convenient, we may in the equation at the end of (6), in which it is to be now assumed that $x_0^2 + y_0^2 = 1$, $x_1^2 + y_1^2 = 1$, write $x - \frac{\alpha}{\gamma}$, $y - \frac{\beta}{\gamma}$ for x, y, and

then restore the original coordinates by writing $\frac{x}{a}$, $\frac{y}{b}$, $\frac{x_0}{a}$, &c., for x, y, x_0, &c., and $\frac{\alpha}{b}$, $\frac{\beta}{a}$, $\frac{\gamma}{ab}$ for α, β, γ, these quantities throughout signifying $\alpha = y_0 - y_1$, $\beta = x_1 - x_0$, $\gamma = x_0 y_1 - x_1 y_0$. I however obtained the equation referred to the point T as origin by a different process, as follows:—

9. Starting from the equation at the commencement of (5), I found that the points (x_0, y_0), (x_1, y_1) being on the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the equation could be transformed into the form

$$\frac{\left(\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - 1\right)^2}{(x-x_0)^2 + (y-y_0)^2} = \frac{\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2}{(x-x_1)^2 + (y-y_1)^2},$$

an equation which (not, as the original one, for all values of (x_0, y_0), (x_1, y_1), but) for values of (x_0, y_0), (x_1, y_1), such that $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$, $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$, breaks up into the line AB and a cubic curve.

10. To simplify the transformation, write as before ax, by, ax_0, &c., for x, y, x_0, &c. We have thus to consider the equation

$$\frac{a^2(x-x_0)^2 + b^2(y-y_0)^2}{(xx_0 + yy_0 - 1)^2} = \frac{a^2(x-x_1)^2 + b^2(y-y_1)^2}{(xx_1 + yy_1 - 1)^2},$$

where $x_0^2 + y_0^2 = 1$, $x_1^2 + y_1^2 = 1$, and which equation, I say, breaks up into the line $L = 0$, and into a cubic.

Write for shortness $\alpha = y_0 - y_1$, $\beta = x_1 - x_0$, $\gamma = x_0 y_1 - x_1 y_0$, so that the equation of the last mentioned line is $\alpha x + \beta y + \gamma = 0$. Then it may be verified that, in virtue of the relations between (x_0, y_0), (x_1, y_1), we have identically

$$(x-x_0)(xx_1 + yy_1 - 1) + (x-x_1)(xx_0 + yy_0 - 1) = (\alpha x + \beta y + \gamma)\frac{x_0 + x_1}{\alpha\gamma}(\gamma x + \alpha),$$

$$(x-x_0)(xx_1 + yy_1 - 1) - (x-x_2)(xx_0 + yy_0 - 1) = \beta x^2 - \alpha xy - \gamma y - \beta;$$

and, similarly,

$$(y-y_0)(xx_1 + yy_1 - 1) + (y-y_1)(xx_0 + yy_0 - 1) = (\alpha x + \beta y + \gamma)\frac{y_0 + y_1}{\beta\gamma}(\gamma y + \beta).$$

$$(y-y_0)(xx_1 + yy_1 - 1) - (y-y_1)(xx_0 + yy_0 - 1) = \beta xy - \alpha y^2 + \gamma x + \alpha.$$

11. The equation in question may be written $a^2 P + b^2 Q = 0$, where

$$P = (x-x_0)^2(xx_1 + yy_1 - 1)^2 - (x-x_1)^2(xx_0 + yy_0 - 1)^2,$$
$$Q = (y-y_0)^2(xx_1 + yy_1 - 1)^2 - (y-y_1)^2(xx_0 + yy_0 - 1)^2,$$

values which are given by means of the formulæ just obtained; there is a common factor $\alpha x + \beta y + \gamma$ which is to be thrown out; and we have also, as is at once verified, $\frac{y_0 + y_1}{\beta} = \frac{x_0 + x_1}{\alpha}$, so that these equal factors may be thrown out. We thus obtain the cubic equation

$$a^2(\gamma x + \alpha)(\beta x^2 - \alpha xy - \gamma y - \beta) + b^2(\gamma y + \beta)(\beta xy - \alpha y^2 + \gamma x + \alpha) = 0.$$

This is simplified by writing $x - \frac{\alpha}{\gamma}$ for x, $y - \frac{\beta}{\gamma}$ for y. It thus becomes

$$a^2x\left[\gamma x - \alpha(\beta x - \alpha y) - \gamma^2 y\right] + b^2 y\left[\gamma y - (\beta x - \alpha y) + \gamma^2 x\right] = 0;$$

or, what is the same thing,

$$a^2x\left[\gamma x(\beta x - \alpha y) - \alpha\beta x + (\alpha^2 - \gamma^2)y\right] + b^2 y\left[\gamma y(\beta x - \alpha y) - (\beta^2 - \gamma^2)x + \alpha\beta y\right] = 0;$$

that is, $\gamma(a^2x^2 + b^2y^2)(\beta x - \alpha y) + a^2\left[-\alpha\beta x^2 + (\alpha^2 - \gamma^2)xy\right]$

$$+ b^2\left[-(\beta^2 - \gamma^2)xy + \alpha\beta y^2\right] = 0.$$

12. Restoring $\frac{x}{a}, \frac{x_0}{a}, \frac{x_1}{a}$ for x, x_0, x_1, and $\frac{y}{a}, \frac{y_0}{a}, \frac{y_1}{a}$ for y, y_0, y_1; writing, consequently, $\frac{\alpha}{b}, \frac{\beta}{a}, \frac{\gamma}{ab}$ in place of α, β, γ, if α, β, γ are still used to denote $\alpha = y_0 - y_1$, $\beta = x_1 - x_0$, $\gamma = x_0y_1 - x_1y_0$, the equation becomes

$$\gamma(x^2 + y^2)\left[b^2\beta x - a^2\alpha y\right] + a^2\left[-b^2\alpha\beta x^2 + (a^2\alpha^2 - \gamma^2)xy\right]$$

$$+ b^2\left[-(b^2\beta^2 - \gamma^2)xy + a^2\alpha\beta y^2\right] = 0,$$

where now, as originally, $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$, $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$; viz., this is the equation, referred to the point T as origin, of the locus of the point P considered as the intersection of tangents from A, B to the variable confocal conic; and it is consequently the equation which would be obtained as indicated in (8). The locus is thus a circular cubic; the equation is identical in form with that obtained (2) for the locus of the point at which A, T and B, T subtend equal angles, and the complete identification of the two equations may be effected without difficulty.

13. I may remark that M. Chasles has given (Comptes Rendus, tom. 58, February, 1864) the theorem that the locus of the intersections of the tangents drawn from a fixed conic to the conics of a system (μ, ν) is a curve of the order 3ν. The confocal conics, *qua* conics touching four fixed lines, are a system $(0, 1)$; hence, taking for the fixed conic the two points A, B, we have, as a very particular case, the foregoing theorem, that the locus of the intersections of the tangents drawn from two fixed points to a variable confocal conic is a cubic curve.

2778. (Proposed by Professor SYLVESTER.)—Suppose μ any quantity fractional or integral, and write

$$c_0 = a_0;\quad c_1 = \mu a_1,\quad c_2 = \frac{\mu(\mu+1)}{2}a_2,\quad c_3 = \frac{\mu(\mu+1)(\mu+2)}{2.3}a_3,\quad \ldots;$$

also let n be a positive integer such that μ and $\mu + n$ have not different signs; then it has been proved elsewhere, and may be assumed to be true, that the number of changes of sign in the series

$$a_0^2,\quad a_1^2 - a_0a_2,\quad a_2^2 - a_1a_3,\quad \ldots$$

is *an inferior limit* to the number of imaginary roots in the equation

$$c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \ldots = 0.$$

1. Required to prove that the above theorem remains true when we write

$$c_0 = a^0, \quad c_1 = \frac{\mu}{\nu} a_1, \quad c_2 = \frac{\mu(\mu+1)}{\nu(\nu+1)} a_2, \quad c_3 = \frac{\mu(\mu+1)(\mu+2)}{\nu(\nu+1)(\nu+2)} a_3, \quad \ldots,$$

provided ν is a positive integer, and $\mu-\nu+1$, $\mu+\nu$ do not differ in sign.

2. Ascertain whether ν being an integer is essential to the truth of the extended theorem.

Solution by the PROPOSER.

Let $\quad fx = c_0x^n + c_1x^{n-1} + \ldots, \quad Fx = x^{n+1}\phi x + fx,$

$$\phi x = \epsilon x^{\nu-1} + \mu\theta\epsilon^2 x^{\nu-2} + \frac{\mu(\mu+1)}{1.2}\epsilon^3 x^{\nu-3} + \frac{\mu(\mu+1)(\mu+2)}{1.2.3}\theta\epsilon^4 x^{\nu-4} + \ldots,$$

the alternately appearing and disappearing factor θ being any quantity different from unity, ϵ an infinitesimal of like sign with c_0, ν any positive integer, and μ any quantity (fractional or integer) such that μ and $\mu+n+\nu$ do not differ in sign.

Let $\quad c_0 = \frac{\mu(\mu+1)\ldots(\mu+\nu-1)}{1.2\ldots\nu} a_0; \quad c_1 = \frac{\mu(\mu+1)\ldots(\mu+\nu)}{1.2\ldots(\nu+1)} a_1; \quad \ldots.$

Then, by the Postulate of the question, the number of imaginary roots in Fx is not less than the sum of the changes of sign in the two series

$$\epsilon^2;\ (\theta^2-1)\epsilon^4;\ (1-\theta^2)\epsilon^6;\ \ldots.;\ (\ldots.);\ a^2;$$

$$a^2;\ a_1^2 - a_0a_2;\ \ldots.;\ a_n^2.$$

In each of these series, the term a^2, properly speaking, has attached to itself another term, but which may be rejected as of only infinitesimal magnitude. In the penultimate term of the upper series, the parenthesis left vacant (similarly neglecting an infinitesimal term of higher order than the one retained) will be $-\epsilon^{\nu-1}a_0$ or $-\theta\epsilon^{\nu-1}a_0$, according as ν is even or odd. Thus the number of changes of sign in this series is ν if ν is even, and $\nu-1$ if ν is odd.

Again, the roots of Fx consist of a group of infinitely great and a group of finite magnitudes. The former are the roots of the equation $x\phi x + c_0 = 0$, or, which is ultimately the same thing, of $\epsilon x^\nu + c_0 = 0$; the latter of $fx = 0$. The number of imaginary roots of the infinite group is therefore ν or $\nu-1$ according as ν is even or odd; *i. e.*, is the same as the number of changes of sign in the upper series. Hence the number of imaginary roots belonging to $fx=0$ cannot be less than the number of sign-changes in the lower series; but this last number will not be affected if we multiply each quantity, $a_0, a_1, a_2, \ldots.$, by the common factor $\frac{\mu(\mu+1)\ldots(\mu+\nu-1)}{1.2\ldots\mu}$, or, which is the same thing, if we write

$$c_0 = a_0; \quad c_1 = \frac{\mu+\nu}{\nu+1} a_1; \quad c_2 = \frac{(\mu+\nu)(\mu+\nu-1)}{(\nu+1)(\nu+2)} a_2; \quad \ldots.$$

Let now $\nu+1 = \nu'$, $\mu+\nu = \mu'$, so that

$$\mu = \mu' - \nu = \mu' - \nu' + 1, \quad \text{and} \quad \mu+n+\nu = \mu'+n.$$

Then, suppressing the accents, we see that, ν being any positive integer, and μ not intermediate between $\nu-1$ and $-n$, if

$$c_0 = a_0; \quad c_1 = \frac{\mu}{\nu} a_1; \quad c_2 = \frac{\mu(\mu+1)}{\nu(\nu+1)} a_2; \quad \ldots.$$

the number of imaginary roots in fx is not less than that of sign-changes in

$$a_0^2;\ a_1^2-a_0a_2;\ a_2^2-a_1a_3;\ \ldots.$$

When $\nu=1$ and $\mu=-n$, we fall back again upon Newton's rule. One feels strongly inclined (on more than one ground) to suspect that ν need not be integer, but I do not see my way to a proof.

As a corollary, it may be noticed that the equation

$$x^n+\frac{\mu}{\nu}x^{n-1}+\frac{\mu(\mu+1)}{\nu(\nu+1)}x^{n-2}+\ldots.,$$

or more generally, the real function (where θ^2 does not exceed unity)

$$\left(x^n+\frac{\mu(\mu+1)}{\nu(\nu+1)}x^{n-2}+\frac{\mu(\mu-1)(\mu+2)(\mu+3)}{\nu(\nu+1)(\nu+2)(\nu+3)}x^{n-4}+\ldots.\right)$$
$$+\theta\left(\frac{\mu}{\nu}x^{n-1}+\frac{\mu(\mu+1)(\mu+2)}{\nu(\nu+1)(\nu+2)}x^{n-3}+\ldots.\right),$$

when ν is a positive integer, and $\mu-\nu+1$, $\mu+n$ have the same sign (*i. e.* when μ is not intermediate between $\nu-1$ and $-n$), cannot have more than one real root.

N.B.—It is worthy of notice, that if we make fx become the *middle part* of a function of x having for its roots a group of infinitely greats, a group of infinitely smalls, and the roots of fx itself, we may, by a method of demonstration similar to that here employed, pass direct from Newton's rule (as given by Newton) to the final extension of it contained in the question above, but with the difference that, so far as the force of such demonstration takes effect, μ as well as ν will have to be integral, a restriction we know to be superfluous. This is one of the reasons for doubting the necessity of ν being an integer.

For the proof of the Postulate, see the Proceedings of the London Mathematical Society, No. II. The above demonstration would have been just as good if only the first powers of the infinitesimal ϵ, instead of the ascending scale, had been used in ϕx throughout; but I have not thought it worth while to disturb the text by making this simplification, which the reader can do for himself.

To find the number of permutations of n *things taken* r *together.*

By C. R. Rippin, M.A.

Let $P(n, r)$ denote the number. Now, if n things be arranged in all possible ways r at a time, each will stand first as often as the remaining $n-1$ can be arranged $r-1$ at a time, that is, $P(n-1, r-1)$ times; but the whole number of permutations is secured by making each thing stand first as often as possible; and therefore, since there are n things, we must have

$$P(n, r) = n\,.\,P(n-1, r-1).$$

Similarly $P(n-1, r-1) = (n-1)\,.\,P(n-2, r-2)$,

&c. = &c.

$$P(n-r+2, 2) = (n-r+2)\,.\,P(n-r+1, 1) = (n-r+2)(n-r+1.)$$

Multiply and simplify, then we obtain

$$P(n, r) = n(n-1)(n-2)\ldots.(n-r+2)(n-r+1).$$

2798. (Proposed by J. J. WALKER, M.A.)—To find the condition that a given point (x, y, z) may be exterior to the given conic

$$S \equiv (A, B, C, F, G, H)(x, y, z)^2 = 0.$$

Solution by F. D. THOMSON, M.A.

If a point be without a conic, its polar cuts the conic in two real points. Now let $\lambda x + \mu y + \nu z = 0$ be the equation to polar of x, y, z.

Then, for the points of intersection of the polar with the conic, we have the equation

$$(A, B, C, F, G, H)(-(\mu y + \nu z), \lambda y, \lambda z)^2 = 0.$$

Expanding this as a quadratic in $\frac{y}{z}$, and writing down the condition for *real* roots, we have

$$(F^2 - BC, G^2 - CA, H^2 - AB, AF - GH, BG - FH, CH - FG)(\lambda, \mu, \nu)^2 > 0;$$

or, since

$$\lambda : \mu : \nu = \frac{dS}{dx} : \frac{dS}{dy} : \frac{dS}{dz},$$

$$\left(A', B', C', F', G', H'\right)\left(\frac{dS}{dx}, \frac{dS}{dy}, \frac{dS}{dz}\right)^2 > 0 \ldots\ldots\ldots\ldots (1),$$

where A', B', &c. are the *inverse* coefficients.

The condition (1) reduces to

$$(AF^2 + BG^2 + CH^2 - ABC - 2FGH)S > 0, \quad \text{or} \quad \Delta S > 0 \ldots\ldots (2).$$

In the particular case where $y = z = 0$, $S = Ax^2$ and is positive; therefore the condition becomes $\Delta > 0$.

1912. (Proposed by Professor MANNHEIM.)—If a and b be the points of contact, with a curve of the third class, of a double tangent; and if this tangent be intersected in m, n, p by the three tangents to the curve which can be drawn from any point in the plane, then $\frac{am \cdot an \cdot ap}{bm \cdot bn \cdot bp} = \frac{\rho_a}{\rho_b}$, where ρ_a, ρ_b are the radii of curvature at the points a and b.

Solution by the Rev. J. WOLSTENHOLME, M.A.

Take (x, y, z) for tangential coordinates, and (α, β, γ) for areal coordinates. The tangential equation of a curve of the third class, to which $\alpha = 0$ is a double tangent at the points where it meets $\beta = 0$, $\gamma = 0$, is of the form

$$xyz + Ay^3 + By^2z + Cyz^2 + Dz^3 = 0 \ldots\ldots\ldots\ldots (1).$$

For the tangents through an arbitrary point (p, q, r) we shall have

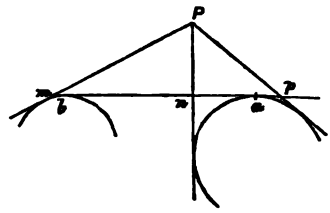
$$px + qy + rz = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

which, combined with (1), leads to the cubic

$$p(Ay^3+By^2z+Cyz^2+Dz^3)-yz(qy+rz)=0;$$

so that $\frac{y_1y_2y_3}{z_1z_2z_3}=\frac{D}{A}$ and is independent of (p, q, r). But the tangent $x_1\alpha+y_1\beta+z_1\gamma=0$ meets the straight line $\alpha=0$ in the point where $y_1\beta+z_1\gamma=0$, or $\frac{am}{bm}=\frac{y_1}{z_1}$; whence $\frac{am\,.\,an\,.\,ap}{bm\,.\,bn\,.\,bp}$ is constant.

To investigate this constant, take a point P near a, from which draw three tangents, and let $\angle Pmn=\delta\phi$, $Ppm=\delta\phi'$. Then the constant required is the limit of $\frac{am\,.\,an\,.\,ap}{bm\,.\,bn\,.\,bp}$ when m moves up to b, and n and p to a,



$$=\text{limit of }\frac{ab\,.\,an\,.\,ap}{\frac{1}{2}\rho_b\delta\phi\,.\,ab\,.\,ab}$$

$$=\frac{2}{ab\,.\,\rho_b}\text{ limit of }\frac{an\,.\,ap}{d\phi};\quad\text{and}\quad\frac{Pp}{\delta\phi}=\frac{Pm}{\delta\phi'}=\frac{ab}{\delta\phi'}$$

$$=\frac{2}{\rho_b}\text{ limit of }\frac{an\,.\,ap}{Pp\,.\,\delta\phi'};$$

and making P approach its limit in such a direction that $an=ap$, we shall therefore have ultimately $\frac{Pp}{an}=1$, and $ap=\frac{1}{2}\rho_a\delta\phi'=\frac{\rho_a}{\rho_b}$.

[A solution by Professor CREMONA has been given on p. 88 of Vol. VII. of the *Reprint*.]

2740. (Proposed by W. S. BURNSIDE, M.A.)—Prove that the envelope of a line given by an equation of the form

$$L\cos 2\phi+M\sin 2\phi+P\cos\phi+Q\sin\phi+R=0$$

may be obtained as the discriminant of a cubic equation, and form this equation.

Solution by the PROPOSER.

The above equation is plainly reducible to the form

$$(a\,b\,c\,f\,g\,h)\,(\cos\phi, \sin\phi, 1)^2=0,$$

which further is equivalent to $(a\,b\,c\,f\,g\,h)\,(\xi, \eta, \zeta)^2=0$ or $U=0$, with $\xi^2+\eta^2-\zeta^2=0$ or $V=0$; whence, if we find the condition that the conics U and V touch (considering $a\,b\,c\,f\,g\,h$ as constants), we shall have the equation of the envelope required. And this is most easily done by forming the discriminant of $U+\lambda V=0$, *sc.* $(ABCD)\,(1, \lambda)^3$, and then equating to zero the discriminant of this cubic.

2636. (Proposed by Professor SYLVESTER.)—
1. If A, B be two confocal conics, and A′ be a third conic having double contact with A, show that there exists a fourth conic B′ having double contact with B and confocal with A′.

2. If A, B be two confocal quadrics, and A′ be a third quadric having continuous contact with A, show that there exists a fourth quadric B′ having continuous contact with B and confocal with A′.

Solution by the Rev. J. WOLSTENHOLME, M.A.

1. Generalized by projection, this amounts to: Given two conics A, B inscribed in the same quadrilateral (of which OO′ is a diagonal); there can be described two conics A′, B′ having double contact with A, B respectively, and such that the tangents to the two from O, O′ are the same; or (to express it more clearly) two of whose common tangents intersect in O and the other two in O′. Reciprocating this, we have two conics a, b passing through four points lying on the straight lines o, o', and the theorem to be proved is that two conics a', b' can be formed having double contact with a, b respectively and meeting the straight lines o, o' in the same points. The simplest case to which this general theorem can be reduced by projection appears to be: Given two conics A, B similar and similarly situated, two circles can be drawn having double contact with A, B respectively, and meeting their common chord in the same two points. This is not at all hard to prove; but as I have not seen how to get a neat solution, I refrain.

2369. (Proposed by H. R. GREER, B.A.)—Given a cubic curve K, and a point on it p; through P is drawn any transversal meeting K again in m and n, and on it is taken a point x such that the anharmonic ratio $(pxmn)$ shall be equal to a given quantity. Prove that the locus of x is a quartic curve with a point of osculation at p, touching K at p (counting for four points of intersection) and in four other points.

Solution by JAMES DALE.

Referring K to rectangular axes passing through p, its equation is of the form $Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 + Fxy + Gy^2 + Hx + Ky = 0$,

which, being transformed to polar coordinates, becomes

$$r^2(A\cos^3\theta + B\cos^2\theta\sin\theta + C\cos\theta\sin^2\theta + D\sin^3\theta) + r(E\cos^2\theta + F\cos\theta\sin\theta + G\sin^2\theta) + H\cos\theta + K\sin\theta = 0.$$

Let now x be a point on any radius vector cutting the curve in m and n, so that the anharmonic ratio $= l : m$; then, putting $px = r$, $pm = r_1$, $pn = r_2$, we get $r\{(l-m)(r_1+r_2) + (l+m)(r_1-r_2)\} = 2(l-m)r_1r_2$.

Substituting the values (r_1+r_2), (r_1-r_2), r_1r_2, obtained from the equation to the curve, and returning to x and y coordinates, we get

$$(l-m)^2\left\{Ex^2+Fxy+Gy^2+2(Hx+Ky)\right\}^2$$
$$=(l+m)^2\left\{Ex^2+Fxy+Gy^2-4(Hx+Ky)(Ax^3+Bx^2y+Cxy^2+Dy^3)\right\}$$

as the equation to the locus of x, which is evidently of the fourth degree.

Let $Ex^2+Fxy+Gy^2+2(Hx+Ky)=0$, which is the equation of the polar conic of p with respect to K, be represented by P_2; and let $Hx+Ky=0$, which is the common tangent at p_1 to P and K, be represented by T_1, and the cubic by K_3; then the locus of x may be written

$$lmP_2^2=2(l+m)^2T_1.K_3;$$

and it may be shown that this curve touches K_3 in the points common to T_1 and P_2, that is, in two coincident points at the origin, T_1 being the common tangent to the three curves, and in four other points.

2494. (Proposed by Professor EVERETT.)—On each side of a hexagon as base, a triangle is described by producing the two adjacent sides to meet, and a second hexagon is formed by joining the vertices of these triangles in order. Show that if either of these two hexagons can be inscribed or circumscribed to a conic, the other can be circumscribed or inscribed to a conic.

2513. (Proposed by Professor EVERETT.)—Let A, B, C, D, E, F be six points. Let BF, CD meet in a; AB, CE in b; AB, CD in c; AF, CE in d; BF, DE in e; and AF, DE in f. Show that, if either of the hexagons ABCDEF, *abcdef* can be inscribed or circumscribed to a conic, the other can be circumscribed or inscribed to a conic.

2539. (Proposed by Professor EVERETT.)—If three conics cut one another so that every two of them have a common chord which is also a chord of a fourth conic, the other three chords of intersection of the three conics meet in a point.

Solution by the PROPOSER.

THEOREM I. [Quest. 2539.] Let S be the fourth conic (*i. e.*, let its equation be S=0), and α, β, γ the three chords common to it and to the others two by two; then $S-l\beta\gamma$, $S-m\gamma\alpha$, $S-n\alpha\beta$ will be the three conics. By subtraction between the two last, we have $m\gamma\alpha-n\alpha\beta$ or $\alpha\left(\frac{\beta}{m}-\frac{\gamma}{n}\right)$, denoting a pair of lines which must be chords of intersection of two of the three conics; and it is obvious that $\frac{\beta}{m}-\frac{\gamma}{n}$, $\frac{\gamma}{n}-\frac{\alpha}{l}$, $\frac{\alpha}{l}-\frac{\beta}{m}$ meet in a point.

THEOREM II. If the three conics break up into pairs of lines, the foregoing theorem has in general, for any given values of α, β, γ, eight distinct applications, that is to say, there will be eight points in each of which three lines meet.

There are two values of l which make $S-l\beta\gamma$ break up into a pair of lines.

For instance, if AC and BD are β and γ, $S - l\beta\gamma$ may denote either AB and DC or AD and BC. We have thus six pairs of lines which we may denote by

$$S - l\beta\gamma, \quad S - m\gamma\alpha, \quad S - n\alpha\beta,$$
$$S - l'\beta\gamma, \quad S - m'\gamma\alpha, \quad S - n'\alpha\beta,$$

and eight points of meeting, viz.,

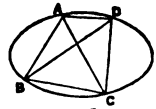


FIG 1.

$$\frac{\alpha}{l} = \frac{\beta}{m} = \frac{\gamma}{n}; \quad \frac{\alpha}{l'} = \frac{\beta}{m'} = \frac{\gamma}{n'}; \quad \frac{\alpha}{l} = \frac{\beta}{m} = \frac{\gamma}{n'}, \quad \&c.,$$

which we may denote by lmn, $l'm'n'$, $l'mn$, $lm'n$, lmn', $lm'n'$, $l'mn'$, $l'm'n$; and the pairs of lines may in like manner be denoted by l, l', m, m', n, n'. Each of the lines passes through two of the points.

Let A, B, C, D, E, F be any six points on a conic, and let α, β, γ denote AD, BE, CF respectively; then

l is BC and EF, m is CD and FA, n is DE and AB;
l' is BF and CE, m' is FD and AC, n' is BD and EA.

$\frac{\beta}{m} - \frac{\gamma}{n}$ will denote the line joining the intersection of AB and CD to that of FA and DE, which we shall indicate by the notation "AB, CD to FA, DE." The eight sets of lines are

AB, CD to FA, DE;	BC, DE to AB, EF;	CD, EF to BC, FA
AC, BD to DF, EA;	BD, CE to EA, FB;	CE, DF to FB, AC
AB, CD to FA, DE;	AB, CE to DE, BF;	AF, CE to CD, BF
BC, DE to AB, EF;	BC, DF to EF, CA;	BA, DF to DE, CA
CD, EF to BC, FA;	CD, EA to FA, DB;	CB, EA to EF, DB
AC, BD to DF, EA;	EF, BD to BC, EA;	EF, AC to BC, DF
BD, CE to EA, FB;	FA, CE to CD, FB;	FA, BD to CD, EA
CE, DF to FB, AC;	AB, DF to DE, AC;	AB, CE to DE, FB

meeting in:

$l\,m\,n'$
$l'm'n$
$l'm\,n'$
$l\,m'n$
$l\,m\,n'$
$l\,m'n$
$l'm\,n'$
$l'm'n$

The figures of the first two cases are subjoined. In both of them, as here lettered, the lines meeting in a point are GL, HM, and KN. It will be observed that the hexagon GHKLMN is formed from ABCDEF, in Fig. 1, in the same way as ABCDEF is formed from GHKLMN, in Fig. 2. A similar relation connects $l'mn$ with $lm'n'$, $lm'n$ with $l'mn'$, and lmn' with $l'm'n$. The following figure, in which the conic may be supposed to pass either through A, B, C, D, E, F, or through a, b, c, d, e, f, applies to the two cases $l'mn$ and $lm'n'$, the capital letters being applicable to the former and the small letters to the latter. In the former case, ad, be, cf will meet in a point; in the latter, AD, BE, CF. We have thus proved that, if either of the two hexagons (mentioned in Questions 2494, 2513) can be inscribed in a conic, the diagonals of the other meet in a point; whence it follows, by the converse of Brianchon's theorem, that this other hexagon can be circumscribed about a conic.

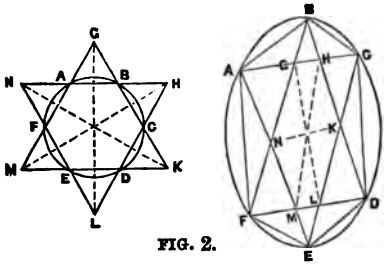


FIG. 2.

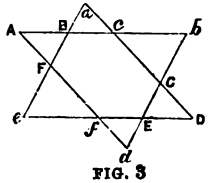


FIG. 3

Note.—Each of the eight sets of lines is connected with a Pascal's line by the relation that, if two triangles be such that the lines joining corresponding vertices meet in a point, the three intersections of corresponding sides will be in the same straight line. In Fig. 2 the two triangles are GKM, LNH. It will be found that the line $\frac{\beta}{m}=\frac{\gamma}{n}$ is the Pascal ABEDCF, and that $\frac{\beta}{m'}=\frac{\gamma}{n'}$ is the Pascal DBEACF.

2101. (Proposed by M. DARBOUX.)—Trouver les conditions nécessaires et suffisantes pour que les quatre racines d'une équation du quatrième degré forment un quadrilatère inscriptible. Trouver la surface et le rayon de ce quadrilatère.

Solution by J. J. WALKER, M.A.

If (a, b, c, d) be the four real and positive roots of a biquadratic $x^4-px^3+qx^2-rx+s=0$, it is evident that the condition of its being possible to construct a quadrilateral figure the four sides of which shall be represented in length by these roots is, simply, that the greatest of these values should be less than the sum of the other three; *i. e.*, that the expression

$$u=(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d)$$

should be positive. In terms of the coefficients of the biquadratic

$$u\equiv 2\Sigma a^2b^2-\Sigma a^4+8abcd\equiv 4\Sigma a^2b^2-(\Sigma a^2)^2+8abcd$$
$$=4(q^2-2pr+2s)-(p^2-2q)^2+8s=p^2(4q-p^2)+8(2s-pr).$$

This condition being fulfilled, let a, b be two adjacent sides of the figure, θ the angle between them, and δ the length of the diagonal joining their ends; and let θ' be the angle between the remaining sides c, d. Then, if the figure is to be inscribable in a circle of radius R, we have

$$\cos\theta+\cos\theta'=0, \quad \text{whence} \quad \delta^2=\frac{(a^2+b^2)cd+(c^2+d^2)ab}{ab+cd}; \quad \text{and}$$

$$\cos\theta=\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}, \quad \text{whence} \quad \sin^2\theta=\frac{4(ab+cd)^2-\left\{a^2+b^2-(c^2+d^2)\right\}^2}{4(ab+cd)^2}$$
$$=\frac{2\Sigma a^2b^2-\Sigma a^4+8abcd}{4(ab+cd)^2}=\frac{p^2(4q-p^2)+8(2s-pr)}{4(ab+cd)^2}.$$

The area of the quadrilateral $=\frac{1}{2}(ab+cd)\sin\theta$

$$=\tfrac{1}{4}\left\{p^2(4q-p^2)+8(2s-pr)\right\} \quad \ldots\ldots\ldots (2).$$

Again $$R^2=\frac{\delta^2}{4\sin^2\theta}=\frac{\left\{(a^2+b^2)cd+(c^2+d^2)ab\right\}(ab+cd)}{u}.$$

This numerator is a symmetric function of a, b, c, d, being equal to $\Sigma a^3bcd+\Sigma a^2b^2c^2=(p^2-2q)s+r^2-2qs=(p^2-4q)s+r^2$, which is, plainly, essentially positive. Hence we obtain

$$R^2=\frac{(p^2-4q)s+r^2}{p^2(4q-p^2)+8(2s-pr)} \quad \ldots\ldots\ldots\ldots (3).$$

In the case of a convex quadrilateral, then, the only condition to be fulfilled by the real and positive roots of the biquadratic is $u=0$; and such a figure is always inscribable in a real circle by giving its angles the values determined by $\sin\theta = \frac{\sqrt{u}}{2(ab+cd)}$.

If the quadrilateral is to be *skew* or *twisted*,

$$\beta^2 = \frac{(a^2+b^2)\,cd-(c^2+d^2)\,ab}{cd-ab};$$

and that this should be positive, $\{(a^2+b^2)\,cd-(c^2+d^2)\,ab\}\,(cd-ab)$ must be positive; or $\Sigma a^2b^2c^2-\Sigma a^3bcd$, *i. e.*, $r^3-2qs-(p^2-2q)\,s$, or r^3-p^2s, must be positive. If this condition and $u=0$ are satisfied, it will be found that

$$\sin^2\theta = \frac{4(ab-cd)^2-(a^2+b^2-c^2-d^2)^2}{4(ab-cd)^2} = \frac{p(4pq-p^3-8r)}{4(ab-cd)^2};$$

$$\text{Area} = \tfrac{1}{4}p(4pq-p^3-8r); \text{ and } R^2 = \frac{r^2-p^2s}{p(4pq-p^3-8r)}.$$

2616. (Proposed by C. W. MERRIFIELD, F.R.S.)—Let two intersecting tetrahedra have all their edges bisected by the same system of Cartesian axes, each axis through two opposite edges of each tetrahedron; then the solid about the origin has the origin for its centre of figure.

Solution by the PROPOSER.

The origin is of course the centre of figure of each tetrahedron separately. Moreover each tetrahedron is completely defined by the angles between the axes, and the intercepts of the edges, except that it may be inverted.

Calling the intercepts a, b, c, and writing the equation $\frac{x}{p}-\frac{y}{q}+\frac{z}{r}=1$ in the abbreviated form $p\,\bar{q}\,r$, the equations of the sides of one tetrahedron are

$$a\,b\,c,\quad a\,\bar{b}\,\bar{c},\quad \bar{a}\,b\,\bar{c},\quad \bar{a}\,\bar{b}\,c,$$

while those of the inverse tetrahedron are

$$\bar{a}\,\bar{b}\,\bar{c},\quad \bar{a}\,b\,c,\quad a\,\bar{b}\,c,\quad a\,b\,\bar{c}.$$

The two systems of planes form an octohedron.

I begin with the clearest case of intersection, in which the corners of the inner tetrahedron simply project through the sides of the outer one, without crossing its edges.

Let the planes of the tetrahedra be represented by the following groups of planes, opposite planes (not necessarily parallel) written under one another,

$$\begin{array}{llll} a\,b\,c, & a\,\bar{b}\,\bar{c}, & \bar{a}\,b\,\bar{c}, & a\,b\,\bar{c}, \\ \bar{l}\,\bar{m}\,\bar{n}, & \bar{l}\,m\,n, & l\,\bar{m}\,n, & l\,m\,\bar{n}. \end{array}$$

The secondary pyramid made by the plane $\bar{l}\,\bar{m}\,\bar{n}$ with the three planes $a\,\bar{b}\,\bar{c}$, $\bar{a}\,b\,\bar{c}$, $\bar{a}\,\bar{b}\,c$ has its four points $(x_0y_0z_0)$, &c. determined by the four systems of equations

$$\left.\begin{matrix} a\,\bar{b}\,\bar{c} \\ \bar{a}\,b\,\bar{c} \\ \bar{a}\,\bar{b}\,c \end{matrix}\right\} \quad \left.\begin{matrix} \bar{l}\,\bar{m}\,\bar{n} \\ \bar{a}\,b\,\bar{c} \\ \bar{a}\,\bar{b}\,c \end{matrix}\right\} \quad \left.\begin{matrix} \bar{l}\,\bar{m}\,n \\ \bar{a}\,\bar{b}\,c \\ a\,\bar{b}\,\bar{c} \end{matrix}\right\} \quad \left.\begin{matrix} \bar{l}\,\bar{m}\,\bar{n} \\ \bar{a}\,b\,\bar{c} \\ a\,\bar{b}\,\bar{c} \end{matrix}\right\}$$

which give the following values;—

$$x_0 = -a, \qquad y_0 = -b, \qquad z_0 = -c,$$

$$x_1 = -a, \qquad y_1 = \frac{bmn\,(a-l)}{l\,(bn+cm)}, \qquad z_1 = \frac{cmn\,(a-l)}{l\,(bn+cm)},$$

$$x_2 = \frac{aln\,(b-m)}{m\,(an+cl)}, \qquad y_2 = -b, \qquad z_2 = \frac{cln\,(b-m)}{m\,(an+cl)},$$

$$x_3 = \frac{alm\,(c-n)}{n\,(am+bl)}, \qquad y_3 = \frac{blm\,(c-n)}{n\,(am+bl)}, \qquad z_3 = -c.$$

The other secondary pyramids are obtained from the first by changing simultaneously the signs of any two of $a\,b\,c$ and of the corresponding pair of $l\,m\,n$. Now, changing the signs of $a\,b\,l\,m$ and leaving the signs of $c\,n$ unchanged is equivalent to changing the signs of x and y and retaining those of z and of all the constants, and so with regard to the other two permutations. Hence it is clear that the sum of the sixteen x's is identically equal to zero, and so of the y's and z's. It follows, then, that if these sixteen points be considered as the centres of spheres of equal weight, the centre of gravity is at the origin. If, therefore, the four secondary pyramids have all the same volume, the proposition will be proved.

Consider the hexahedron (or parallelepiped)

$$a\,0\,0, \quad 0\,\bar{b}\,0, \quad 0\,0\,c,$$
$$\bar{a}\,0\,0, \quad 0\,\bar{b}\,0, \quad 0\,0\,\bar{c},$$

the twelve diagonals of the faces of which are the edges of the two tetrahedra $(a\,b\,c)$ and $(\bar{a}\,\bar{b}\,\bar{c})$. Let us suppose that its alternate corners are cut off by the four planes $\bar{l}\,\bar{m}\,\bar{n}$, $\bar{l}\,m\,n$, $l\,\bar{m}\,n$, $l\,m\,\bar{n}$.

These planes will cut off equal lengths upon the corresponding edges of the hexahedron, and therefore the pyramids cut off will be of equal volume, as may easily be shown. Next, if we consider each plane of the type $(l\,m\,n)$ as the common base of two pyramids, one cut off from the hexahedron, and the other from the original tetrahedron $(a\,b\,c)$, we see that these have the same altitude, and that the proportion between their bases is the same at all the four corners. Hence the pyramids cut off from the tetrahedron are also all equal *inter se*. But this being the case, it has already been shown that their centre of figure is at the origin. Hence the centre of figure of the solid about the origin, that is, of the remainder of the tetrahedron $(a\,b\,c)$, is also at the origin.

I have chosen the simplest case for demonstration. The general proposition may be inferred from the principle of the permanence of equivalent forms, or, with a little more trouble in picking out the details, may be proved independently.

Cor. 1.—The solid formed about the origin by any number of such intersecting tetrahedra, and by hexahedra of the same system, has also the origin for its centre of figure.

Cor. 2.—Cleavage parallel to one of the coordinate planes simply shifts the centre of figure along the corresponding coordinate axis.

Cor. 3.—If the solid be made up of any number of tetrahedra and hexahedra about the same axes and having their corners on the same four dia-

gonals, then the effect of cleavage in a tetrahedral plane is simply to shift the centre of figure along the diagonal corresponding to the cleavage plane. If there are to be no hollow spaces, there must of course be at most only one hexahedron, and one direct and one inverse tetrahedron.

Octohedral cleavage is thus seen to be more complex than hexahedral, as might be expected.

NOTE ON QUESTION 2740: BY PROFESSOR CAYLEY.

The envelope of the curve

$$A\cos 2\theta + B\sin 2\theta + C\cos\theta + D\sin\theta + E = 0,$$

(where A, B, C, D, E are any functions of the coordinates, and θ is a variable parameter,) is obtained in the particular case $E=0$ (Salmon's *Higher Plane Curves*, p. 116), and the same process is applicable in the general case where E is not $= 0$. From the great variety of the problems which depend upon the determination of such an envelope, the result is an important one, *and ought to be familiarly known* to students of analytical geometry. We have only to write $z = \cos\theta + i\sin\theta$, the trigonometrical functions are then given as rational functions of z, and the equation is converted into a quartic equation in z; the result is therefore obtained by equating to zero the discriminant of a quartic function. The equation, in fact, becomes

$$A\frac{1}{2}\left(z^2+\frac{1}{z^2}\right) + B\frac{1}{2i}\left(z^2-\frac{1}{z^2}\right) + C\frac{1}{2}\left(z+\frac{1}{z}\right) + D\frac{1}{2i}\left(z-\frac{1}{z}\right) + E = 0,$$

that is, $$A(z^4+1) - Bi(z^4-1) + C(z^3+z) - Di(z^3-z) + 2Ez^2 = 0;$$

or, multiplying by 12 to avoid fractions, this is

$$(a, b, c, d, e)(z, 1)^4 = 0,$$

where $$a = 12(A-Bi),\quad b = 3(C-Di),\quad c = 4E,$$
$$e = 12(A+Bi),\quad d = 3(C+Di);$$

and substituting in

$$(ae-4bd+3c^2)^3 - 27(ace-ad^2-b^2e+2bcd-c^3)^2 = 0,$$

the equation divides by 1728, and the final result is

$$\left\{12(A^2+B^2) - 3(C^2+D^2) + 4E^2\right\}^3$$
$$- \left\{27A(C^2-D^2) + 54BCD - [72(A^2+B^2) + 9(C^2+D^2)]E + 8E^3\right\}^2 = 0.$$

It is to be noticed, that in developing the equation according to the powers of E, the terms in E^6, E^4 each disappear, so that the highest power is E^3; the degree in the coordinates, or order of the curve, is on this account sometimes lower than it would otherwise have been.

2823 (Proposed by Professor SYLVESTER.)—Show that on a chess-board the chance of a rook moving from one square to another without changing

colour is $\frac{3}{7}$; but that without altering the equality of the *number* of the black and white squares, but only the manner of their distribution, the chance may be made equal to $\frac{1}{2}$.

I. *Solution by* MORGAN JENKINS, M.A.

Wherever the rook may be placed on the board, there are in each of the two lateral directions in which the piece can move 3 squares of the same and 4 of a different colour to that of the square on which the rook stands. The required chance is therefore $\frac{3}{7}$.

No possible distribution of the colours can be made which shall raise the chance to $\frac{1}{2}$ *wherever the rook may be placed*; for if x be the number of black squares in a row, then out of the 14 squares to which the rook may move 7 squares must be of the same and 7 of different colour to that of the square occupied by the rook. Hence, placing the rook on any of the x black squares, we should have $x-1$ other black squares in the same row, and therefore $8-x$ other black squares in the same column; and placing the rook on any of the $8-x$ white squares, we should have x squares in the same row, and therefore $7-x$ black squares in the same column, whence follows the equation

$$x(9-x)+(8-x)(7-x)=32 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The only solution of this is $x=4$; which shows that there could be only 4 black squares in any row, and in like manner in any column; but this is inconsistent with the supposition in (1), that if there are x or 4 black squares in a row, there will be $9-x$ or 5 in some of the columns.

Hence the chance cannot be $\frac{1}{2}$ for every position of the rook; but by supposing that every position of the rook is equally probable, we may, by various distributions of the colours, make the total chance for all positions of the rook $\frac{1}{2}$.

Let $x_1, x_2, \ldots x_8$ be the number of black squares in the rows,

$y_1, y_2, \ldots y_8$ " " " columns;

then $x_1+x_2+\ldots+x_8=32$, $y_1+y_2+\ldots+y_8=32$ .....(2, 3);

and also the following equation is easily obtainable,

$$\left.\begin{array}{l} x_1(x_1-1)+x_2(x_2-1)+\ldots+(8-x_1)(7-x_1)+\ldots \\ +y_1(y_1-1)+y_2(y_2-1)+\ldots+(8-y_1)(7-y_1)+\ldots \end{array}\right\} = 32\times 14,$$

which reduces to $x_1^2+\ldots+y_1^2+\ldots=16\times 18$ .....(4).

It is easily seen that, if $x_1, x_2, \ldots, y_1, y_2, \ldots$ be a set of solutions, then $8-x_1, 8-x_2, \ldots, 8-y_1, 8-y_2, \ldots$ is another set, which may be called a *complementary set.*

To facilitate the finding of solutions of (2), (3), and (4), I have *made the assumption* (involving possibly a loss of generality) that

$$x_1^2+x_2^2+\ldots+x_8^2=16\times 9=y_1^2+y_2^2+\ldots+y_8^2 \ldots\ldots (5, 6).$$

Supposing that there are a rows containing 1 black square, b rows containing 2 black squares, and so on, we have

$$a+b+c+\ldots=8,\ a+2b+3c+\ldots=32,\ a+4b+9c+\ldots=144 \ldots. (7, 8, 9);$$

whence we obtain

$$b+3c+6d+10e+\ldots=56,\ c+3d+6e+\ldots=32 \ldots. (10, 11).$$

The following sets of values of $x_1, x_2, \ldots . x_8$ will be found, I think, to be all which can be obtained from the positive integral solutions of (7), (10), and (11).

[6, 6, 4, 4, 4, 4, 2, 2]	self-complementary
[6, 6, 5, 4, 3, 3, 3, 2]	[6, 5, 5, 5, 4, 3, 2, 2]
[7, 5, 4, 4, 4, 3, 3, 2]	[6, 5, 5, 4, 4, 4, 3, 1]
[7, 5, 5, 3, 3, 3, 3, 3]	[5, 5, 5, 5, 5, 3, 3, 1].

The following squares are specimens of distributions obtainable from these sets of values :—

4	W	W	W	W	B	B	B	B
4	W	W	W	W	B	B	B	B
2	W	W	W	W	B	B	W	W
2	W	W	W	W	B	B	W	W
4	B	B	B	B	W	W	W	W
4	B	B	B	B	W	W	W	W
6	B	B	B	B	B	B	W	W
6	B	B	B	B	B	B	W	W
	4	4	4	4	6	6	2	2

2	W	W	W	W	W	B	W	B
4	B	W	W	W	W	B	B	B
3	B	W	W	W	W	W	B	B
3	B	B	B	W	W	W	W	W
3	B	B	B	W	W	W	W	W
5	B	B	B	B	B	W	W	W
6	B	B	B	B	B	W	B	W
6	B	B	B	B	B	B	W	W
	7	5	5	3	3	3	3	3

[Professor SYLVESTER remarks that any anallagmatic distribution of the black and white squares will satisfy the condition. For, in such a square, whether the motion is from a row to a row or from a column to a column, there will by its definition be as many passages from like colour to like as from a colour to its opposite. Suppose an anallagmatic square is made isochromatic, *i. e.*, having always the same difference (positive or negative) between the black and white colours in each row and column (see the diagram at the end of Vol. X. of the *Reprint*). Then, putting n for the number of squares in a side, and x, $n-x$ for the number of the separate colours respectively, it is clear from the above solution that we must have

$$\Sigma x(n-x) = \tfrac{1}{2}\Sigma n \cdot \frac{n-1}{2}, \quad \therefore\ x(n-x) = \frac{n^2-n}{4}, \quad \text{i. e., } x = \frac{n \pm \sqrt{n}}{2},$$

which proves that an anallagmatic square cannot be made isochromatic unless the number of places it contains is a perfect fourth power. This disposition springs directly from an improved form given to NEWTON'S complete rule for limiting the total number of imaginary roots in an algebraic equation.]

II. *Solution by* MATTHEW COLLINS, B.A.

1. As each line and column of a chessboard contains 4 white and 4 black squares, therefore upon whatever square a rook be placed there are 3 other squares of the same colour, and 7 other squares altogether in the line or column in which the rook can move, and hence his chance of moving into a square of the same colour as that he left is plainly $= \frac{3}{7}$.

2. If the line bisecting two opposite sides of the chessboard be the boundary between the white and the black squares, it is plain the square common to any line and column has, upon that line and column taken jointly, 10 other squares of the same colour as itself; and as there are 14 other squares altogether upon that line and column, hence in this case the chance of a rook moving into a square of the same colour as that it left is $= \frac{10}{14} = \frac{5}{7}$.

3. If the diagonal divides the white from the black squares, and if the squares through which it passes be white and black alternately, the required chance of a rook not changing colour when moved would plainly be

$$\frac{14.7+10.9+6.11+2.13}{32.14} = \frac{5}{8}.$$

4. But if the four white squares on the diagonal occupy half of it, the other half running through four black squares, then the required chance is

$$\frac{1}{32.14}(4.6+6.7+6.8+5.9+5.10+3.11+2.12+1.13) = \frac{279}{448}.$$

5. Lastly, if one row consist of eight white squares, and another row of eight black squares, the remaining six rows of the chess-board being left unchanged, the required chance is then plainly

$$\frac{8.10+24.6}{32.14} = \frac{1}{2}.$$

2741. (Proposed by S. TEBAY, B.A.)—A moveable event (depending on the moon) happened in the year y; show that it will happen again in the year $y+19(28t-5a+8b)$; where a may be 0, 1, 2, 3; b the number of completed centuries since the year y which are not leap years; and t any arbitrary integer.

Solution by the PROPOSER.

Let $Y = y+4n-a$ be the year; then the number of days in this interval is $365(4n+a)+n-b = 7(208n+52a)+5n+a-b$;

therefore $5n+a-b$ must be divisible by 7;

therefore $n = 7m-3a+3b$, and $4n+a = 28m-11a+12b$,

which must be divisible by 19;

therefore $m = 19t-3a+5b$, $\therefore\ 4n+a = 19(28t-5a+8b)$,

and $Y = y+19(28t-5a+8b)$.

Example: Easter-day cannot happen later than April 25th; the last time this took place was in the year 1734; when will it happen again? Let $t=0$, $a=0$, $b=1$; therefore $Y = 1886$, the year required.

2598. (Proposed by G. M. SMITH, B.A.)—A uniform rod (mass $= \mu$) is placed inside a spherical shell (mass $= m$, radius $= a$) which, under the influence of gravity, rolls down the surface of a rough sphere (radius $= r$); find equations for the movements of the rod and shell, there being no friction

between them; and hence show that, if a particle μ is placed near the lowest point of a spherical shell (m) which performs small oscillations on a rough horizontal plane, the length of the simple equivalent pendulum is $\frac{2m}{2m+\mu}\,a$.

Solution by the PROPOSER.

Let the horizontal and vertical lines through the centre O of the sphere be taken as axes of x and y respectively; let (x, y) be the coordinates of the centre C of the shell; ψ = angle through which the shell has turned; ϕ = angle between radius and Oy; $\theta = COy$; c = perpendicular from C on rod. Then

$$m\left(\frac{d^2x}{dt^2}\,\delta x + \frac{d^2y}{dt^2}\,\delta y + a^2\frac{d^2\psi}{dt^2}\,\delta\psi\right) + \mu\cdot\left\{\left(\frac{d^2x}{dt^2} - c\frac{d^2\cos\phi}{dt^2}\right)\left(\delta x - c\delta\cos\phi\right)\right.$$
$$\left. + \left(\frac{d^2y}{dt^2} - c\frac{d^2\sin\phi}{dt^2}\right)\left(\delta y - c\delta\sin\phi\right) + k^2\frac{d^2\phi}{dt^2}\,\delta\phi\right\}$$
$$= -(m+\mu)\,g\,\delta y + \mu c g\cos\phi\,\delta\phi.$$

Substituting for x and y their values in terms of θ, and selecting the coefficients of $\delta\theta$ and $\delta\phi$, we obtain, by equating these coefficients separately to zero, the equations

$$(2m+\mu)(a+r)\frac{d^2\theta}{dt^2} - \mu c\left\{\cos(\phi-\theta)\frac{d^2\phi}{dt^2} - \sin(\phi-\theta)\left(\frac{d\phi}{dt}\right)^2\right\}$$
$$+ (m+\mu)\,g\cos\theta = 0,$$

$$c(a+r)\left\{\cos(\phi-\theta)\frac{d^2\theta}{dt^2} + \sin(\phi-\theta)\left(\frac{d\theta}{dt}\right)^2\right\} - (k^2+c^2)\frac{d^2\phi}{dt^2} + gc\cos\phi = 0.$$

In the particular case in which the radius becomes a particle, and the sphere a horizontal plane, these equations become

$$(2m+\mu)\,a\frac{d^2\psi}{dt^2} + \mu c\frac{d^2\cos\phi}{dt^2} = 0,$$

$$ac\sin\phi\frac{d^2\psi}{dt^2} - (k^2+c^2)\frac{d^2\phi}{dt^2} + cg\sin\phi = 0,$$

which give (by putting $\phi = \frac{1}{2}\pi - \beta$, where β is small)

$$\frac{\mu a^2}{\mu+2m}\cdot\frac{d^2\beta}{dt^2} - a^2\frac{d^2\beta}{dt^2} = ga\beta, \quad \text{or} \quad \frac{2m}{2m+\mu}\cdot\frac{d^2\beta}{dt^2} = g\beta,$$

showing that the length of the simple pendulum $= \frac{2m}{2m+\mu}\,a$.

In the case in which the sphere reduces to a plane of inclination $= i$, our equations become

$$(2m+\mu)\,a\frac{d^2\psi}{dt^2} + \mu c\frac{d^2\cos(\phi+i)}{dt^2} + (m+\mu)\,g\sin i = 0,$$

$$ac\sin(\phi+i)\frac{d^2\psi}{dt^2} - (k^2+c^2)\frac{d^2\phi}{dt^2} + gc\cos\phi = 0.$$

2221. (Proposed by W. S. BURNSIDE, M.A.)—1. Show how to determine the locus of the feet of perpendiculars from a fixed point on the generating lines of a system of confocal quadrics.

2. Prove that this locus is described by foci of the plane sections passing through the fixed point.

Solution by the PROPOSER.

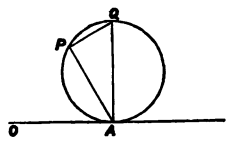
1. Since the foot of a perpendicular on a generator is the foot of a perpendicular on a tangent plane through that generator, the question (1) is reduced to first finding the locus of feet of perpendiculars on tangent planes to one surface of the system, and then eliminating the arbitrary parameter which enters by means of the equation of the surface itself.

2. Consider the section of a surface of the system by a plane P, and let one of the foci of the section be at F; this being so, reciprocating origin at F, the section of the reciprocal surface by the plane P is a circle; consequently the section of the asymptotic cone of this surface is a circle; but the asymptotic cone is similar and similarly placed to the cone generated by lines passing through F perpendicular to the tangent planes through F to the given surface; therefore one of the focal lines of this cone is the perpendicular to the plane P at F. Finally the focal lines of the tangent cone are the generators of the confocal hyperboloid of one sheet through the vertex.

2795. (Proposed by S. TEBAY, B.A.)—Find the average square of the distance between the centres of the inscribed and circumscribed circles of a triangle inscribed in a given circle.

I. *Solution by the* Rev. J. WOLSTENHOLME, M.A.

If a be the radius of the given circle, x radius of an inscribed circle, square of distance between the centres is $a^2 - 2ax$, and the question is equivalent to finding the average value of x. Now, considering one point A of the triangle fixed, which is clearly allowable; AO the tangent at A; P, Q the other angular points of the triangle,



$$\angle \mathrm{OAP} = \theta, \quad \angle \mathrm{OAQ} = \phi, \quad (\phi > \theta), \quad x = 4a \sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}(\phi - \theta) \cos \tfrac{1}{2}\phi,$$

$$\therefore \text{ average value of } x = a \frac{\int_0^{\pi} \int_{\theta}^{\pi} 4 \sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}(\phi - \theta) \cos \tfrac{1}{2}\phi \, d\theta \, d\phi}{\int_0^{\pi} \int_{\theta}^{\pi} d\theta \, d\phi}$$

$$= a \frac{6 - \frac{1}{2}\pi^2}{\frac{1}{2}\pi^2} = a\left(\frac{12}{\pi^2} - 1\right)$$

(See *Book of Mathematical Problems*, Ex. 1036);

therefore average square of distance between the centres is $3a^2\left(1 - \frac{8}{\pi^2}\right)$.

II. *Solution by the* PROPOSER.

Let D be the distance between their centres; then

$$\begin{aligned} D^2 &= R^2 - 2Rr \\ &= R^2(1 - 8\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C) \\ &= R^2 - 4R^2\sin\tfrac{1}{2}A\left\{\sin\tfrac{1}{2}(A+2B) - \sin\tfrac{1}{2}A\right\}. \end{aligned}$$

Supposing the vertical angle to remain constant while B varies from 0 to $\pi - A$, we find the sum of the squares of the distances

$$= 2\int_0^{\pi}\int_0^{\pi-A} D^2\,dA\,dB = 3R^2(\pi^2 - 8).$$

The number of triangles $= 2\int_0^{\pi}\int_0^{\pi-A} dA\,dB = \pi^2.$

Hence the average square of the distance $= 3R^2\left(1 - \frac{8}{\pi^2}\right).$

2797. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—In any complete quadrilateral, the radical axis of the three circles whose diameters are the diagonals will pass through the centres of perpendiculars of the four triangles formed by the four straight lines.

Solution by MATTHEW COLLINS, B.A.

Let ABCD (Fig. 1) be the quadrilateral; then the circles whose diameters are the three diagonals AC, BD, EF will have their centres at the middle points G, H, K of these diagonals; the three altitudes of the triangle ABF, viz., AOA′, BOB′, FOF′, meet in one point O (its ortho-centre); therefore Gg, perpendicular to AOA′, is parallel to BC and bisects AA′ in g; similarly, Hh perpendicular to BB′, and Kk perpendicular to FF′, bisect BB′ and FF′ in h and k.

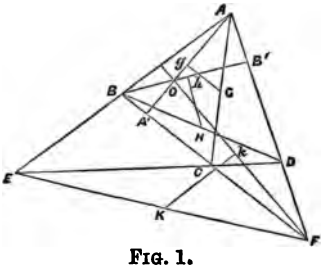


FIG. 1.

Now, the square of the tangent drawn from O to the circle (G) $= OG^2 - AG^2 = Og^2 - Ag^2 = OA \cdot OA'$; similarly, the square of the tangent from O to the circle (H) $= OH^2 - BH^2 = Oh^2 - Bh^2 = OB \cdot OB'$, and the square of the tangent from O to the circle (K) $= OK^2 - FK^2 = Ok^2 - Fk^2 = OF \cdot OF'$.

But $OA \cdot OA' = OB \cdot OB' = OF \cdot OF'$, therefore the tangents from O to the three circles are equal to each other, consequently the ortho-centre O of the triangle ABF is a point on the radical axis of these three circles. For like reasons, the ortho-centres of the other three triangles formed by three

other sides of the quadrilateral ABCD (viz., the triangles ADE, BCE, CDF) lie also upon the same radical axis.

COROLLARY.—As the three circles (G), (H), (K) are proved coaxal, their centres, viz., the middle points of the three diagonals of the complete quadrilateral ABCDEF, lie in one straight line. A simple proof of this theorem may also be obtained as follows:—

Let ABCDEF (Fig. 2) be the complete quadrilateral; AC, BD, EF its three diagonals. Through B, C, D, E, F draw lines parallel to the sides of the angle A, forming the parallelograms ABGD, AEHF, &c.

Then (by Euclid I. 43), parallelogram CK = AC = CK', and therefore parallelogram LK = L'K'; hence (by Euclid I. 43, *ex absurdo*) the points C, G, H lie in one straight line.

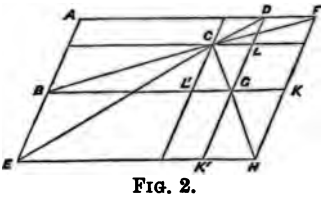


FIG. 2.

Now, as the diagonals of a parallelogram bisect each other, the middle point of the diagonal BD is the same as the middle point of AG; and, for a like reason, the middle point of the third diagonal EF is the same as the middle point of AH. But the middle points of AC, AG, AH lie plainly in one straight line parallel to CGH; therefore the middle points of the three diagonals AC, BD, EF of the complete quadrilateral ABCDEF lie in one straight line.

NOTE.—Several new and curious theorems can be easily obtained from the foregoing theorem and its corollary by means of the methods of *reciprocation* and *inversion*.

2622. (Proposed by A. W. PANTON, B.A.)—The equation connecting the distances (r_1, r_2, r_3) of any point on a Cartesian oval from the foci is

$$(\beta-\gamma)\,\alpha^{\frac{1}{2}} r_1 + (\gamma-\alpha)\,\beta^{\frac{1}{2}} r_2 + (\alpha-\beta)\,\gamma^{\frac{1}{2}} r_3 = 0,$$

where α, β, γ are the distances of the foci from the triple focus.

Solution by W. S. MCCAY, B.A.

The Cartesian oval may be written

$$lr_1 + mr_2 + nr_3 = 0, \text{ or } \lambda r_1 + \mu r_2 = d \ldots\ldots\ldots\ldots (1, 2).$$

Now to reduce the first of these to the second by means of the identity

$$Ar_1^2 + Br_2^2 + Cr_3^2 = D \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

(where A, B, C are the distances between the foci, and D = ABC) we get by substituting r_3 in (3) from (1)

$$Ar_1^2 + Br_2^2 + C\left(\frac{lr_1 + mr_2}{n}\right)^2 = D.$$

That this may be identical with (2), we must have

$$\frac{l^2}{A}+\frac{m^2}{B}+\frac{n^2}{C}=0, \quad \text{also} \quad \lambda^2:\mu^2=\frac{An^2+Cl^2}{Bn^2+Cm^2} \ldots\ldots\ldots (5,6).$$

The last equation becomes by means of (5) $\lambda^2:\mu^2=\dfrac{A^2}{l^2}:\dfrac{B^2}{m^2}$ .....(7).

Now since (2) may be written

$$\left(\lambda^2 r_1^2-\mu^2 r_2^2-\frac{\lambda^2+\mu^2}{\lambda^2-\mu^2}\right)^2+\frac{4d^2\lambda^2\mu^2}{\lambda^2-\mu^2}\left(r_1^2-r_2^2+\frac{d^2}{\lambda^2-\mu^2}\right)=0,$$

it follows that if (α, β, γ) are the distances of the foci from the triple focus

$$\left(\text{centre of circle } \lambda^2 r_1^2-\mu^2 r_2^2=\frac{\lambda^2+\mu^2}{\lambda^2-\mu^2}\right), \qquad \lambda^2\alpha-\mu^2\beta=0.$$

Hence (7) becomes $\quad l:m=A\alpha^{\frac{1}{2}}:B\beta^{\frac{1}{2}};$

so too $\quad m:n=B\beta^{\frac{1}{2}}:C\gamma^{\frac{1}{2}};$

therefore $\quad l:m:n=(\beta-\gamma)\alpha\ :(\gamma-\alpha)\beta^{\frac{1}{2}}:(\alpha-\beta)\gamma^{\frac{1}{2}}.$

2670. (Proposed by C. T. HUDSON, LL.D.)—An observer, seated in the aisle of a cathedral and looking westward, sees the horizontal lines above the arches converging to a point before him, and consequently cutting the horizon at a certain angle. On his facing northwards, he sees that the same lines are now parallel to the horizon. Required the curve that they will appear to lie in, as he gradually turns his head through 90°.

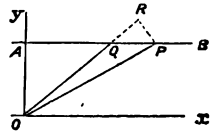
Solution by the PROPOSER.

This problem is indeterminate; for the plane on which the given lines are seen projected at any instant, will at that instant be referred to various distances according to the fancy of each observer.

(i.) Let this plane be supposed to be vertical, and always at a fixed distance from the observer; then the ultimate intersections of its various positions will be a vertical cylinder of which the axis passes through the observer, and a plane through his eye and the given horizontal lines will cut the cylinder in an ellipse, which therefore is the required curve.

(ii.) Let that portion of the given horizontal line which is seen distinctly at any instant be supposed to be referred to its real distance from the observer at that instant.

Let O be the observer, AB the given horizontal line, Oy and Ox lines perpendicular and parallel to AB in the plane drawn through AB and O, QP the portion of AB which is seen distinctly when the observer is looking along OP, but which is seen projected into the position PR at right angles to OP. Let also $AO=c$, $PO x=\theta$; then the equation to PR is $\quad x\cos\theta+y\sin\theta=c\operatorname{cosec}\theta;$

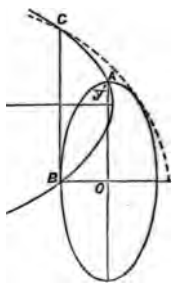


and the ultimate intersections of all its positions obtained by varying θ lie in the curve $x^2 = 4c(c-y)$ ..... (1).

The sensation of the curvature of AB is, according to this hypothesis, produced by the attention being fixed on the extremity of the line RP, which always moves so as to touch the parabola (1), and at the same time remain at right angles to the visual ray.

2762. (Proposed by the EDITOR.)—Find the envelope of an ellipse which has one vertex in the curve of a given parabola, and touches at its adjacent vertex the extremity of a fixed double-ordinate of the parabola.

I. *Solution by* J. J. WALKER, M.A.



Taking the ordinate of the parabola and the axis of the ellipse which passes through the vertex of contact as axes of y and x, the equation to the parabola will be (n being the fixed ordinate)

$$y^2 - 2ny + px = 0 \quad \ldots\ldots\ldots\ldots (1),$$

and that of the ellipse

$$\frac{y^2}{(y'+n)^2} + \frac{p^2x^2}{(y'^2-n^2)^2} + \frac{2px}{y'^2-n^2} = 0 \quad \ldots (2),$$

where y' is the ordinate of the parabola passing through the first vertex of the ellipse, since the axes of the ellipse will plainly be $y'+n$ and $\frac{n^2-y'^2}{p}$. Equation (2), when multiplied by $(y'^2-n^2)^2$ and arranged by powers of y', becomes

$$(y^2+2px)\,y'^2 - 2ny^2y' + n^2y^2 + p^2x^2 - 2pn^2x = 0,$$

the discriminant of which, with respect to y' as variable, is

$$(y^2+2px)(n^2y^2+p^2x^2-2pn^2x) - n^2y^4, \quad \text{or} \quad p^2x^2(y^2+2px-4n^2).$$

Hence the required envelope is the parabola

$$y^2 + 2px - 4n^2 = 0,$$

which has as ordinate the fixed double ordinate, and a parameter double that of the given parabola.

II. *Solution by the* REV. J. WOLSTENHOLME, M.A.; S. BILLS; *and others.*

The equation of the parabola being $y^2 = 4ax$, let the coordinates of the fixed vertex be $a\alpha^2$, $2a\alpha$, and of the moving vertex $a\lambda^2$, $2a\lambda$; then the equation of the ellipse is $\frac{(x-a\lambda^2)^2}{a^2(\alpha^2-\lambda^2)^2} + \frac{(y-2a\alpha)^2}{4a^2(\alpha-\lambda)^2} = 1,$

or $$4(x-a\lambda^2)^2 + (\alpha+\lambda)^2(y-2a\alpha)^2 = 4a^2(\alpha^2-\lambda^2)^2,$$

or $$4(x-a\alpha^2)(x+a\alpha^2-2a\lambda^2)+(a+\lambda)^2(y-2a\alpha)^2=0;$$

and the envelope is

$$a^2(y-2a\alpha)^4=\left\{(y-2a\alpha)^2-8a(x-a\alpha^2)\right\}\left\{a^2(y-2a\alpha)^2+4(x^2-a^2\alpha^4)\right\},$$

or $$(x-a\alpha^2)^2\left\{(y-2a\alpha)^2-8a(x+a\alpha^2)\right\}=0.$$

The required envelope is thus seen to consist of the fixed double ordinate, and a parabola similarly situated to the given one, of twice the dimensions, and having its vertex in the diameter through the given vertex. For the contact to be real, the vertices must be on opposite sides of the axis of the parabola.

2800. (Proposed by J. WILSON.)—Three circles passing through a point P form a circular triangle ABC, and each side of this circular triangle or its continuation is cut orthogonally at the point P′ by a circle passing through P; prove that the three circles described about the triangles PAP′, PBP″, PCP‴ are co-axal.

Solution by ARCHER STANLEY.

By inverting the figure with respect to the point P, it will readily be found that the theorem is the inverse of the well-known one, that the perpendiculars are concurrent which are drawn from the vertices of a triangle to the respectively opposite sides.

2688. (Proposed by J. GRIFFITHS, M.A.)—A variable triangle circumscribes an equilateral hyperbola, and is such that its *nine-point* circle passes through the centre of this curve; prove that the locus of the centre of its *circumscribing* circle is the hyperbola in question.

Solution by the Rev. J. WOLSTENHOLME, M.A.

If an equilateral hyperbola be inscribed in a triangle, the centre of the hyperbola lies on the polar circle of the triangle: if, then, the centre lies on the nine-point circle, it must lie on the circumscribed circle, since the three circles are coaxal. Let O be this centre, ABC the triangle, and let the asymptotes of the hyperbola meet the circumscribed circle in P, Q. Then, since OPQ, ABC are triangles in the same conic (circle), their sides will touch one conic; but five of them touch the hyperbola, therefore also the sixth (PQ) touches the hyperbola: it will be bisected in the point of contact, which will therefore be the centre of the circumscribed circle of the triangle. Hence, if a rectangular hyperbola touch the sides of a triangle, and its centre lie on the nine-point circle of the triangle, the centre of the circumscribed circle lies on the hyperbola; which proves the theorem.

2654. (Proposed by W. S. BURNSIDE, M.A.)—Determine the polar reciprocal of the quartic curve

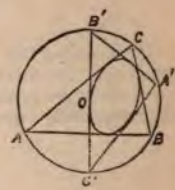
$$y^2z^2+z^2x^2+x^2y^2+2xyz(lx+my+nz)=0.$$

Solution by the PROPOSER.

Write $\frac{1}{X}, \frac{1}{Y}, \frac{1}{Z}$ for x, y, z, and the quartic is transformed to $X^2+Y^2+Z^2+2lYZ+2mZX+2nXY=0$, and the polar of a point $\alpha\beta\gamma$ becomes $\alpha YZ+\beta ZX+\gamma XY=0$. So the question is reduced to finding when these two conics touch, which is easily done.

2726. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—If a conic touch the sides of a triangle and pass through the centre of the circumscribed circle, the director circle of the conic will touch the circumscribed circle of the triangle.

Solution by W. S. MCCAY, B.A.



If the conic be $(lx)^{\frac{1}{2}}+(my)^{\frac{1}{2}}+(nz)^{\frac{1}{2}}=0$, it follows from the general equation of the director circle given at p. 338 of Salmon's *Conics*, that the radical axis of director circle and circumscribing circle of the triangle is $l\cot A\,.\,x+m\cot B\,.\,y+n\cot C\,.\,z=0$;

and if this touch the circumscribing circle $(ayz+bxz+cxy=0)$, we must have

$$l^2\cos^2 A+m^2\cos^2 B+n^2\cos^2 C-2m^2n^2\cos B\cos C-2l^2n^2\cos B-2m^2l^2\cos^2 C=0,$$

or $$(l\cos A)^{\frac{1}{2}}+(m\cos B)^{\frac{1}{2}}+(n\cos C)^{\frac{1}{2}}=0.$$

But this is the condition that centre of circumscribing circle (cos A, cos B, cos C) should lie on the conic.

[Mr. WOLSTENHOLME'S proof is as follows:—

Let O be the centre of the circumscribed circle, and let a tangent to the conic at O meet the circle in B′, C′; then the tangents from B′, C′ to the conic will meet in a point (A′) which will lie on the circle, since ABC, A′B′C′ are triangles circumscribing the same conic. These tangents being at right angles, A′ is a point on the director circle; and since B′C′ is bisected in O, OA′ passes through the centre of the conic, that is, through the centre of the director circle. The circles therefore touch at A′.]

2770. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—If a conic be drawn through S, S′, the foci (both real or both impossible) of a given conic, and osculate the conic at a point P, the tangents at S, S′ will intersect in the centre of curvature at P.

Solution by the PROPOSER.

The equation of any conic touching the given conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(a\cos\theta,\ b\sin\theta)$ is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\left(\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}-1\right)\left(\mathrm{L}\cos\theta\frac{x}{a}+\mathrm{M}\sin\theta\frac{y}{b}-\mathrm{N}\right)=0;$$

if this osculate at the point θ, $\quad \mathrm{L}\cos^2\theta+\mathrm{M}\sin^2\theta=\mathrm{N}$;

if it also pass through the two foci on the axis of x,

$$\mathrm{L}+\mathrm{N}=0,\quad \left(1-\frac{b^2}{a^2}\right)(1+\mathrm{L}\cos^2\theta)=1-\mathrm{N};$$

Also the pole of SS′ is given by the equations

$$2(1+\mathrm{L}\cos^2\theta)\frac{x}{a}+\frac{\sin\theta\cos\theta}{b}(\mathrm{L}+\mathrm{M})y=0,\quad \frac{\mathrm{M}+\mathrm{N}}{b}y\sin\theta+2(1-\mathrm{N})=0.$$

But, from the former equations,

$$-\mathrm{L}=\mathrm{N}=\frac{b^2}{a^2\sin^2\theta+b^2\cos^2\theta}=\frac{\mathrm{M}\sin^2\theta}{1+\cos^2\theta};$$

therefore $\quad \frac{\mathrm{M}+\mathrm{N}}{1-\mathrm{N}}=\frac{2b^2}{(a^2-b^2)\sin^4\theta}$, $\quad$ therefore $\quad y=-\frac{a^2-b^2}{b}\sin^3\theta$;

and

$$\frac{\mathrm{L}+\mathrm{M}}{1+\mathrm{L}\cos^2\theta}=\frac{2b^2\cos^2\theta}{a^2\sin^4\theta},$$

therefore $\quad \frac{x}{a}=\frac{b\cos^3\theta}{a^2\sin^3\theta}\cdot\frac{a^2-b^2}{b}\sin^2\theta=\frac{a^2-b^2}{a^2}\cos^3\theta$;

or the tangents at S, S′ intersect in the point $\left(\frac{a^2-b^2}{a}\cos^3\theta,\ \frac{b^2-a^2}{b}\sin^3\theta\right)$, the centre of curvature at P.

It may be noticed that, in any conic passing through SS′ and touching at P, the tangent and normal at P, form with PS, PS′ a harmonic pencil; therefore SS′ and the normal at P are conjugates with repect to this conic, or the tangents at S, S′ intersect on the normal at P.

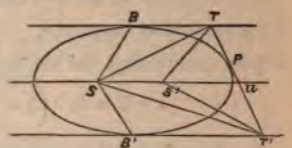
2799. (Proposed by A. MARTIN.)—If n dice are thrown, what is the chance of an odd number of aces turning up?

Solution by the REV. J. WOLSTENHOLME, M.A.

The chance is obviously $\frac{6^n-4^n}{2\times 6^n}$. If the dice be regular tetrahedra, the chance is $\frac{1}{2}-\frac{1}{2^{n+1}}$.

2789. (Proposed by W. S. M'CAY, B.A.)—If two circles pass through the foci of an ellipse and touch the same variable tangent to the curve, the angle at which they intersect is constant and equal to $2\tan^{-1}\frac{b}{c}$.

Solution by the Rev. J. WOLSTENHOLME, M.A.



Let SS′ be foci of an ellipse, TPT′ a tangent meeting the tangents at BB′, the ends of the minor axis in TT′, and the major axis in U. Then $\angle UTS' = \angle BTS = \angle TSS'$; therefore triangles UTS′, UST are similar, and UT : US′ = US : UT, or US . US′ = UT²; therefore the circle through STS′ will touch the tangent at T, so the circle through ST′S′ will touch the tangent at T′; and the angle at which these circles cut in S, S′ is the sum of the angles STS′, ST′S′, and therefore the difference of the angles TS′T′, TST′.

But $\angle TST' = \frac{1}{2}\angle BSB'$, and $\angle TS'T' = \frac{1}{2}$ external angle $BS'B' = \pi - \frac{1}{2}BSB'$;

therefore angle at which the circles cut $= \pi - BSB' = 2\tan^{-1}\frac{c}{b}$.

The first part of this problem I have already set in the form, "If a circle pass through the foci of an ellipse, and common tangents be drawn to the ellipse and circle, their points of contact with the circle will lie on the tangents to the ellipse at the ends of the minor axis."

2780. (Proposed by Professor HIRST.)—The envelope of the chord common to an ellipse and its circle of curvature is a curve of the fourth clas which has three double tangents, one at infinity and the two others coin dent with the conjugate diameters equally inclined to the axis. Prove t examine the curve, and consider the cases of the hyperbola and parabola

Solution by J. J. WALKER, M.A.

The equation to the ellipse being $b^2x^2 + a^2y^2 = a^2b^2$, if the coord any point on it be $a\cos\theta$, $b\sin\theta$, the reciprocals of the intercepts the axes by the chord common to the ellipse and the circle of cu

this point will be $\alpha = \frac{\cos\theta}{a\cos 2\theta}$, β [illegible] $\frac{\sin\theta}{2\theta}$, and eliminating

these equations, $a^2\alpha^2 + b^2\beta^2 = (a^2\alpha$ [illegible] is the envelope of th curve of the fourth class, havin [illegible] tangents the line points $a\alpha \pm b\beta = 0$ with the centr [illegible] equal conjugat also the line $\alpha = 0$, $\beta = 0$.

For the hyperbola, the sign of double tangents become imaginary.

In the case of the parabola $y^2 =$ envelope of the chord common to it and

is similarly found to be $a = 3m\beta^2$, a parabola having the same vertex with the given parabola, and as its axis the production of axis of the other. Its parameter is three times that of the given parabola.

The equations to the above envelopes in Cartesian coordinates are

$$(b^2x^2 + a^2y^2 - 4a^2b^2)^3 + 27\,(b^2x^2 - a^2y^2) = 0 \quad \text{and} \quad y^2 + 12mx = 0$$

respectively, the equations to the ellipse and parabola being $b^2x^2 + a^2y^2 = a^2b^2$ and $y^2 = 4mx$ respectively. From the former it appears that the points of contact of the equi-conjugate diameters lie on a similar and coaxal ellipse, and that there are cusps at these points.

[The equation of the envelope has been given by Mr. WOLSTENHOLME, in his Solution of Question 2579 (2), (*Reprint*, Vol. X. p. 91) in the neat form

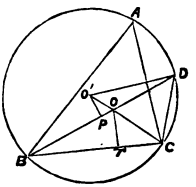
$$\left(\frac{x}{a} + \frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{x}{a} - \frac{y}{b}\right)^{\frac{2}{3}} = 2,$$

which has cusps at the points $\frac{x^2}{a^2} = \frac{y^2}{b^2} = 2$, and of course two at infinity.

The form of the equation to the envelope given by Mr. WALKER in Question 2834 may be readily deduced from Mr. WOLSTENHOLME'S.]

To express the distance between the centres of the circumscribed and inscribed circles in terms of the radii of those circles.
By J. WALMSLEY, B.A.

Let O′, O be centres of circumscribed and inscribed circles respectively of the triangle ABC. Join O′O, OB, OC. Produce BO to D, and join DO′. Draw O′P perpendicular to BD, and therefore bisecting it.



Then $$BO = \frac{r}{\sin \frac{1}{2}B},$$

and $$DCO = DCA + ACO = DBA + ACO$$
$$= OBC + OCB = DOC;$$

therefore $$DO = DC = \frac{a \sin \frac{1}{2}B}{\sin A} = 2R \sin \tfrac{1}{2}B.$$

Hence $$O'D^2 - O'O^2 = DP^2 - OP^2 = DO \cdot OB = 2Rr.$$

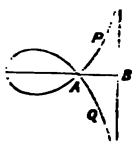
therefore $$O'O^2 = R^2 - 2Rr.$$

In an analogous manner, if O_1 be centre and r_1 radius of escribed circle touching BC, we may show that $O'O_1^2 = R^2 + 2Rr_1$.

2776. (Proposed by W. K. CLIFFORD, B.A.) — Through A, the double point of a circular cubic, draw AB perpendicular to the asymptote; if chords be drawn to the curve subtending a right angle at the double point, show that there is a fixed point in AB at which also they subtend a right angle.

Solution by the REV. J. WOLSTENHOLME, M.A.

The equation may be taken $y^2 = x^2\frac{x+a}{b-x}$, the origin being the double point. If (x_1y_1), (x_2y_2) be the ends of two chords AP, AQ at right angles to each other, $x_1x_2+y_1y_2=0$; and



$$y_1^2 = x_1^2\frac{x_1+a}{b-x_1}, \quad y_2^2 = x_2^2\frac{x_2+a}{b-x_2};$$

therefore $(x_1+a)(x_2+a) = (b-x_1)(b-x_2)$, therefore $x_1+x_2 = b-a$.

But the equation of the circle on PQ is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$, or $x^2+y^2-x(x_1+x_2)-y(y_1+y_2)=0$;

therefore, when $y=0$, $x=0$ or $x=x_1+x_2=b-a$,

giving a fixed point on AB at which PQ subtends a right angle.

2544. (Proposed by Professor BALL.)—There are two values of d for which α, β, γ, the roots of the cubic $ax^3+3bx^2+3cx+d=0$ fulfil the linear relation $A\alpha+B\beta+C\gamma=0$. Show how to find the quadratic equation of which these values of d are the roots.

I. *Solution by the* REV. J. WOLSTENHOLME, M.A.

The relations $\alpha+\beta+\gamma = -\frac{3b}{a}$, $A\alpha+B\beta+C\gamma=0$, $\beta\gamma+\gamma\alpha+\alpha\beta = \frac{3c}{a}$,

lead at once to a quadratic equation in γ; whence

$$\gamma_1+\gamma_2 = -\frac{3b}{a}\frac{A^2+B^2-C(A+B)}{A^2+B^2+C^2-BC-CA-AB} \equiv 2m,$$

$$\gamma_1\gamma_2 = \frac{3ac(A-B)^2+9b^2AB}{a^2(A^2+\ldots-BC-\ldots)} \equiv n.$$

But $-(d_1+d_2) = +a(\gamma_1^3+\gamma_2^3)+3b(\gamma_1^2+\gamma_2^2)+3c(\gamma_1+\gamma_2)$,

and $d_1d_2 = \gamma_1\gamma_2(a\gamma_1^2+3b\gamma_1+3c)(a\gamma_2^2+3b\gamma_2+3c)$,

which can be immediately expressed in terms of m and n.

II. *Solution by* J. J. WALKER, M.A.

By the terms of the question,

$$B\beta+C\gamma = -A\alpha \ldots\ldots (1), \quad \alpha\beta+\alpha\gamma = -\alpha\alpha-3b \ldots\ldots (2),$$
$$\alpha\beta\gamma+\alpha(\beta+\gamma)\alpha = 3c \ldots\ldots (3).$$

From (1) and (2), $\alpha(B-C)\beta = \alpha(C-A)\alpha+3bC \ldots\ldots (4)$,

$$\alpha(C-B)\gamma = \alpha(B-A)\alpha+3bB \ldots\ldots (5);$$

$$-a^2(B-C)^2\beta\gamma = a^2(A-B)(A-C)\alpha^2+3ab\left\{2BC-A(B+C)\right\}\alpha+9b^2BC \quad \ldots\ldots\ldots (6).$$

Eliminating β and γ from (2), (3), (6), there results

$$a^2\left\{(A-B)(A-C)-(B-C)^2\right\}\alpha^2+3ab\left\{B^2+C^2-A(B+C)\right\}\alpha$$
$$+3\left\{ac(B-C)^2+3b^2BC\right\} = 0 \quad \ldots\ldots\ldots\ldots\ldots (7);$$

also
$$a\alpha^3+3b\alpha^2+3c\alpha+d = 0 \quad \ldots\ldots\ldots\ldots\ldots (8).$$

The resultant of (7) and (8) is

$$a'^4d^2+9a'\left\{(3b'^2-2a'c')(a'b-ab')-a'b'(a'c-ac')\right\}d$$
$$+81c'^2(a'b-ab')^2+27c'(a'c-ac')\left\{a'(a'c-ac')-3b'(a'b-ab')\right\} = 0 \ldots (9),$$

if we write
$$a' = a^2\left\{(A-B)(A-C)+(B-C)^2\right\},$$
$$b' = ab\left\{B^2+C^2-A(B+C)\right\},$$
$$c' = ac(B-C)+3b^2BC,$$

whence $a'b-ab' = a^2b(A^2+BC)$, $a'c-ac' = a\left\{ac(A-B)(A-C)-3b^2BC\right\}$.

Making these substitutions in (9), it becomes divisible by a^4, and is the required quadratic for d. The coefficients are not capable of any important reduction, as far as I have examined them.

1587. (Proposed by the Rev. T. P. KIRKMAN, F.R.S.)—Apollo and the Muses accepted the challenge of Jove, to vary the arrangement of themselves on their fixed and burnished couches at his evening banquets, till every three of them should have occupied, once and once only, every three of the couches, in every and any order. In how many days, and how many ways, did they accomplish the feat, keeping one arrangement of themselves through all the solutions? Required two or more of these solutions, clearly indicated, so as to save space, by cyclical operations.

Solution by the PROPOSER.

The solution is the positive half of the group of 10.9.8.2 made with 10 elements, which was first given by me in the *Memoirs of the Literary and Philosophical Society of Manchester*, 1861, and whose *title* I gave thus in their *Proceedings* five or six years ago:—

$$10.9.8.2 = 1+180_{82}+144_{5^2}+45_{2^41^2}+80_{31^2}+270_{4^21^2}$$
$$+30_{2^81^4}+180_{81^2}+144_{10}+90_{4^22}+240_{631}+36_{2^5}: \quad Q = 2520.$$

The positive half is the first six terms on the right, comprising, besides unity, the natural order, 180 substitutions having each a circle of 8 and a circle of 2, 144 substitutions having each two circles of 5, 45 having each four

circles of 2, and 2 elements undisturbed, 80 having three circles of 3 and one undisturbed element, &c.

This group of 10.9.8.2, as well as every group so given by its title, can be easily constructed from its title; but it is impossible to state the rules here. It is enough to give one of the 2520 equivalent groups of 10.9.8, thus,

1234567890	1234567890	1234567890
+0324897561	3215648970	1473692580
+3017594862	3126459780	1327984650
+2107864593	4567891230	1742853960
+6280715394	5648972310	1685739240
+3814069275	6459783120	1956278430
+8243907156	7891234560	1869425370
+8639520147	8972315640	1598346720
+4261539078	9783126450	
+2165437809		

under the form (A);G;H where G;H is a group of 9.8, the product of two groups G and H. (A) is no group. The 10.9.8 cannot be written as the product of so few as three groups. We have the group by adding to G;H its nine derivatives by the substitutions of (A). If these be $1, A_1, A_2, \ldots A_9$; $1, g_1, g_2, \ldots g_8$ those of G; and $1 h_1 h_2 \ldots h_7$ those of H; the group

$$(A);G;H = G;H + A_1;G;H + A_2;G;H + \ldots + A_9;G;H,$$

will be found to stand the test, that every product of its substitutions, if $1 = A_0 = g_0 = h_0$, is of the form $A_p;g_q;h_r$. Neither the algebraic nor the tactical demonstration of this can find space here. The group proves itself. These 10.9.8 arrangements are one of the 2520 solutions of the problem which Apollo's party dined out day by day for nearly 5000 years, at some expense of Jove's nectar. There are three, and only three, solutions which have in common the substitutions of G, (A);G;H, (A');G;H', and (A");G;H". No two solutions have more than eight arrangements in common besides unity. The above group (A);G;H is necessary as to the number, and sufficient as to the form, of its arrangements: necessary, because Apollo, Clio, and Urania cannot seat themselves in every way in fewer than 10.9.8 arrangements; and sufficient, because if in Θ and Φ, any pair of arrangements K, L, and M were seated alike, the group would contain the substitution $\frac{\Theta}{\Phi}$, which has KLM undisturbed; but the positive title has no substitution showing three elements undisturbed. From (A);G;H once written out can be formed all other solutions, by the operation $\theta;(A);G;H;\theta^{-1}$, where θ is any substitution not before used. Thus, $\theta = \theta^{-1}$ may be 2134567890 for a second solution.

NOTE.—The unfashionable reader who wishes to learn how to find all possible groups made with n elements, to determine their titles and the number of their equivalents, and from the titles to construct the groups by a direct tactical method, without troubling himself with congruences, has only to fish up, if he can, from a basket in Manchester, a Memoir of mine which has lain there for some years. Our learned societies agree with me in the opinion that it is absurd to waste English paper on such upstart subjects as Groups and Polyedra, which are fit for nothing but make-believe prize-questions of the French Academy. They have done quite enough in placing it beyond dispute in their Proceedings that I have thoroughly discussed both these vast theories, as difficult as they are unvalued.

2847. (Proposed by M. W. CROFTON, F.R.S.)—Prove that

$$\epsilon^{hD^2}.\epsilon^{-kx^2} = (1+4hk)^{-\frac{1}{2}}\epsilon^{-\frac{kx^2}{1+4hk}}, \text{ where } D = \frac{d}{dx}.$$

Solution by J. J. WALKER, M.A.

Expanding the symbolical operator,

$$\epsilon^{hD^2}.\epsilon^{-kx^2} = \left(1 + \frac{hD^2}{1} + \frac{h^2D^4}{1.2} + \ldots.\right)\epsilon^{-kx^2}.$$

But by the known formula for the successive differential coefficients of ϵ^{-kx^2} with respect to x,

$$\epsilon^{kx^2}\ D^0.\epsilon^{-kx^2} = 1,$$

$$\epsilon^{kx^2}\, h\, D^2.\epsilon^{-kx^2} = -\frac{2.1}{1}hk + hk^2(2x)^2,$$

$$\epsilon^{kx^2}\, h^2D^4.\epsilon^{-kx^2} = \frac{4\ldots1}{(1.2)^2}h^2k^2 - \frac{4.3}{1.2}h^2k^3(2x)^2 + \frac{h^2k^4}{1.2}(2x)^4,$$

$$\epsilon^{kx^2}\, h^3D^6.\epsilon^{-kx^2} = -\frac{6\ldots1}{(1.2.3)^2}h^3k^3 + \frac{6\ldots3}{(1.2)^2.3}h^3k^4(2x)^2 - \frac{6.5}{1.2.3}h^3k^5(2x)^4 + \frac{h^3k^6}{1.2.3}(2x)^6,$$

$$\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots$$

It will be found on examination that the rth term (or line) of the pth column in the sum of the above developments is equal to

$$(-1)^{r-1}\frac{(4hk^2x^2)^{p-1}}{1.2\ldots(p-1)} \times \frac{(2r+2p-4).(2r+2p-5)\ldots(2p-1).(4hk)^{r-1}}{1.2\ldots(r-1).r\ldots(r+p-2).p.(p+1)\ldots(r-1).(2)^{2r-2}},$$

when r is greater than p; but when r is equal to or less than p, the denominator of the last factor is $1.2\ldots(r-1).p.(p+1)\ldots(p+r-2).2^{2r-2}$. In both cases this factor is equal to $\frac{(2p-1)(2p+1)\ldots(2p+2r-5)}{1.2\ldots(r-1).2^{r-1}}(4hk)^{r-1}$, which multiplied by $(-1)^{r-1}$ is the rth term in the development of $(1+4hk)^{-\frac{2p-1}{2}}$.

From this it results that $\epsilon^{kx^2}.\epsilon^{hD^2}.\epsilon^{-kx^2} = (1+4hk)^{-\frac{1}{2}}\epsilon^{\frac{4hk^2x^2}{1+4hk}}$; and multiplying both sides of this by ϵ^{-kx^2}, there results for $\epsilon^{hD^2}.\epsilon^{-kx^2}$ the value given in the Question.

[This theorem, which the Proposer has found to possess important applications in the Theory of Errors of Observation, seems deducible in the most direct, though not the most elementary, manner, by putting ϵ^{-kx^2} for $\phi(x)$ in POISSON'S famous transformation (*Traité de Mécanique*, Tom. II., p. 356), which gives

$$\epsilon^{hD^2}\phi(x) = \frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty}\epsilon^{-\omega^2}\phi(x+2h^{\frac{1}{2}}\omega)\,d\omega.]$$

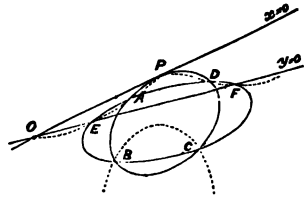
2623. (Proposed by W. S. McCay, B.A.)—If a conic pass through the four points of contact of tangents to a cubic from a point (A) on the curve, and through two other points (B, C) on the cubic; then A is the pole of BC with regard to the conic.

Solution by F. D. Thomson, M.A.

The equation to any cubic may be written

$$S \equiv xV + kyU = 0 \ldots\ldots (1),$$

where U, V are conics intersecting in four points A, B, C, D on the curve, and x and y are straight lines intersecting on the curve, in the point O suppose.



The equation to the polar conic of (1) with reference to the point (x', y', z') is

$$x'\frac{dS}{dx} + y'\frac{dS}{dy} + z'\frac{dS}{dz} = 0,$$

which may be written

$$x'V + ky'U + xP_v + kyP_u = 0 \ldots\ldots\ldots\ldots\ldots (2),$$

where P_u denotes the polar of (x', y', z') with respect to U. But if (x', y', z') be on the line OP which touches U, $x' = 0$, and P_u becomes $x\frac{du}{dx'}$.

Hence the equation (2) reduces to

$$ky'U + x\left(P_v + ky\frac{du}{dx'}\right) = 0.$$

Therefore, if U is itself the polar conic of the point P,

$$P_v + ky\frac{du}{dx'} = 0 \text{ identically.}$$

Therefore P_v coincides with $y = 0$, *i.e.*, with EF, or EF is the polar of P with respect to V.

2606. (Proposed by C. W. Merrifield, F.R.S.)—The developable circumscribing two surfaces of the second degree touches either of them along a curve, which is its intersection with another surface of the second degree.

I. *Solution by the* Rev. R. Townsend, F.R.S.

If U and V be the two quadrics, U′ the polar reciprocal of U with respect to V, and V′ the polar reciprocal of V with respect to U; then, for every plane tangent to both U and V, the point of contact with U being a point on V′, and the point of contact with V being a point on U′, therefore, &c. (See Salmon's *Geometry of Three Dimensions*, 2nd Ed., Art. 207.)

If $a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 = 0$ and $a'\alpha^2 + b'\beta^2 + c'\gamma^2 + d'\delta^2 = 0$ be the equations of U and V referred to their common self-reciprocal tetrahedron, then are

$$\frac{a'^2}{a}\alpha^2+\frac{b'^2}{b}\beta^2+\frac{c'^2}{c}\gamma^2+\frac{d'^2}{d}\delta^2=0 \quad \text{and} \quad \frac{a^2}{a'}\alpha^2+\frac{b^2}{b'}\beta^2+\frac{c^2}{c'}\gamma^2+\frac{d^2}{d'}\delta^2=0$$

those of U′ and V′ referred to the same tetrahedron; which, as is otherwise evident geometrically, is consequently self-reciprocal with respect to the whole four surfaces at once. (See same, Art. 206.)

II. *Solution by the* PROPOSER.

Referring the two surfaces to the centre of one of them, and to axes parallel to conjugate diameters in both, their equations may be written thus:—

$$U = a_1x_1^2+\beta_1y_1^2+\gamma_1z_1^2+\delta_1=0,$$

$$V = a_2(x_2-l)^2+\beta_2(y_2-m)^2+\gamma_2(z_2-n)^2+\delta_2-a_2l^2-\beta_2m^2-\gamma_2n^2=0.$$

The tangent plane of U is $a_1x_1x+\beta_1y_1y+\gamma_1z_1z+\delta_1=0$.

The tangent plane of V is

$$a_2(x_2-l)(x-l)+\beta_2(y_2-m)(y-m)+\gamma_2(z_2-n)(z-n)+\delta_2-a_2l^2-\beta_2m^2-\gamma_2n^2=0,$$

or $a_2(x-l)x+\beta_2(y_2-m)y+\gamma_2(z_2-n)z+\delta_2-a_2lx_2-\beta_2my_2-\gamma_2mz_2=0$.

And, if these are to be identical,

$$\frac{a_1x_1}{a_2(x_2-l)}=\frac{\beta_1y_1}{\beta_2(y_2-m)}=\frac{\gamma_1z_1}{\gamma_2(z_2-n)}=\frac{\delta_1}{\delta_2-a_2(x_2-\beta_2my_2-\gamma_2mz_2)}.$$

Substituting these values of $x_1y_1z_1$ in U, we get

$$\frac{a_2^2\delta_1(x_2-l)^2}{a_1}+\frac{\beta_2^2\delta_1(y_2-m)^2}{\beta_1}+\frac{\gamma_2^2\delta_1(z_2-n)^2}{\gamma_1}+(\delta_2-a_2lx_2-\beta_2my_2-\gamma_2nz_2)^2=0.$$

Hence the developable of two quadrics touches each along a curve through which another quadric can be drawn.

Conversely, the envelope of the tangent planes to a quadric, along its intersection with another quadric, is the developable circumscribing the first and some other quadric.

2804. (Proposed by S. TEBAY.)—A straight pole stands vertically on a slope inclined to the south. If it be broken at random by the wind blowing in a given direction, so that the upper end of the pole rests upon the slope, determine the probable area of the triangle thus formed; and deduce the result for a horizontal plane.

Solution by the PROPOSER.

Let a be the length of the pole, x the length of the portion standing, α the inclination of the slope to the vertical, β the azimuth of the wind measured on the slope, and ϕ the angle between the standing piece and the base of the triangle. Then $\cos\phi=\cos\alpha\cos\beta$, and

$$\Delta=\tfrac{1}{2}x\sin\phi\left\{x\cos\phi+(a^2-2ax+x^2\cos^2\phi)^{\frac{1}{2}}\right\}.$$

The greatest value of x is $\frac{a}{1+\sin\phi}$, which is also a measure of the number of triangles. Hence we find

$$\int \Delta\, dx = \tfrac{1}{2}a^3 \left\{ \frac{\sin\phi\cos\phi}{3(1+\sin\phi)^2} - \frac{\sin\phi}{3\cos^2\phi} + \frac{\sin\phi}{2\cos^4\phi} - \frac{\sin^2\phi}{2\cos^5\phi} \log\cot\tfrac{1}{2}\phi \right\};$$

and therefore the average area of the triangle is

$$\tfrac{1}{2}a^2 \sin\phi\,(1+\sin\phi) \left\{ \frac{\cos\phi}{3(1+\sin\phi)^3} - \frac{1}{3\cos^3\phi} + \frac{1}{2\cos^4\phi} - \frac{\sin^2\phi}{2\cos^5\phi} \log\cot\tfrac{1}{2}\phi \right\}.$$

When $a = \frac{1}{2}\pi$, we have $\phi = \frac{1}{2}\pi$; and expanding $\log\cot\frac{1}{2}\phi = \frac{1}{2}\log\frac{1+\cos\phi}{1-\cos\phi}$, we find the average area $= \frac{1}{15}a^2$.

2618. (Proposed by the Rev. N. M. FERRERS, M.A.)—A pack of n different cards is laid, face downwards, on a table. A person names a certain card, that and all the cards above it are shown to him and removed; he names another, and the process is repeated. Prove that the chance of his naming the top card during the operation is

$$1 - \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} - \frac{1}{\lfloor 4} + \ldots\ldots - \frac{(-1)^n}{\lfloor n}.$$

Solution by PROFESSOR WHITWORTH.

Let P_n represent the chance of the top card being named in the series of operations on n cards.

At the first operation any of the n cards is equally likely to be named.

If the top one is named, what is required is done, and the *à posteriori* chance of success is unity.

If the second be named, there are $n-2$ cards left, and the *à posteriori* chance of success is P_{n-2}.

If the third be named, then the *à posteriori* chance of success is P_{n-3}, and so on.

If the last card be named, the chance is zero; therefore

$$P_n = \frac{1}{n}\left\{1 + P_{n-2} + P_{n-3} + \ldots\ldots + P_2 + P_1\right\},$$

or

$$nP_n = 1 + P_1 + P_2 + \ldots\ldots + P_{n-3} + P_{n-2}.$$

Similarly (writing $n-1$ for n)

$$(n-1)P_{n-1} = 1 + P_1 + P_2 + \ldots\ldots + P_{n-3};$$

therefore, by subtraction, $nP_n - (n-1)P_{n-1} = P_{n-2}$,

or

$$n(P_n - P_{n-1}) = -(P_{n-1} - P_{n-2}).$$

Multiply both sides by $(-1)^n \lfloor n-1$, then

$$(-1)^n \lfloor n\,(P_n - P_{n-1}) = (-1)^{n-1} \lfloor n-1\,(P_{n-1} - P_{n-2}).$$

Now, writing $n-1$, $n-2$, $n-3$, &c. successively for n, we obtain, *ex equali*,

$$(-1)^n \lfloor n\,(P_n - P_{n-1}) = \lfloor 2\,(P_2 - P_1).$$

But it is obvious that $P_1 = 1$ and $P_2 = \frac{1}{2}$, therefore

$$(-1)^n \lfloor n\,(P_n - P_{n-1}) = -1, \quad \text{or} \quad P_n - P_{n-1} = \frac{(-1)^{n-1}}{\lfloor n}.$$

Writing 2, 3, 4, &c. successively for n, we obtain

$$P_2 - P_1 = -\frac{1}{\lfloor 2}, \quad P_3 - P_2 = +\frac{1}{\lfloor 3}, \quad P_4 - P_3 = -\frac{1}{\lfloor 4}, \quad \&c.$$

$$P_n - P_{n-1} = \frac{(-1)^{n-1}}{\lfloor n}.$$

Therefore, by addition,

$$P_n - P_1 = -\frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} - \frac{1}{\lfloor 4} + \ldots\ldots - \frac{(-1)^n}{\lfloor n},$$

or

$$P_n = 1 - \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} - \frac{1}{\lfloor 4} + \ldots\ldots - \frac{(-1)^n}{\lfloor n}.$$

With the notation of Questions 2637, 2648, this chance (of naming a top card) is $1 - e_n^{-1}$, and the chance of not naming a top card is e_n^{-1}.

2813. (Proposed by J. J. WALKER, M.A.)—The equation of a conic referred to an axis and tangent at vertex being $ax^2 + by^2 + 2dx = 0$, if the conic be turned about the vertex in its own plane through a right angle, the locus of the intersection of any tangent in the original position with the same line in the new position of the conic is the (bicircular) quartic

$$b\left\{a(x^2+y^2) + 2d(x+y)\right\}(x^2+y^2) = d^2(x-y)^2,$$

and the corresponding locus for the normal is the sextic

$$b\left\{a(x^2+y^2) + b(x-y)^2\right\}\left\{a(x^2+y^2) + d(x+y)\right\}^2 = (a-b)^2 d^2 (x^2-y^2)^2.$$

What does this latter equation become in the case of the parabola ?

I. *Quaternion Solution by* W. H. LAVERTY, B.A.

1. Taking for simplicity the equation to the conic in the form

$$b^2x^2 + a^2y^2 = 2ab^2x,$$

the vector equations to the tangents are

$$\rho = \alpha + \alpha\cos\tau + \beta\sin\tau + x(-\alpha\sin\tau + \beta\cos\tau),$$

$$\rho = \frac{a}{b}\beta(1+\cos\tau) - \frac{b}{a}\alpha\sin\tau + y\left(\frac{a}{b}\beta\sin\tau + \frac{b}{a}\alpha\cos\tau\right);$$

and that these may coincide, we must have

$$x(a^2\sin^2\tau + b^2\cos^2\tau) = a(a\sin\tau + b\cos\tau) + (a^2-b^2)\sin\tau\cos\tau + ab;$$

whence substituting, and putting $a^2\sin^2\tau + b^2\cos^2\tau = \mathrm{P}$, we have for the required locus

$$\rho\,.\,\mathrm{P} = \left\{ba\,(b\cos\tau - a\sin\tau) + a\beta\,(b\cos\tau + a\sin\tau)\right\}(1+\cos\tau).$$

To transform this to a Cartesian equation, we have

$$\frac{x\,.\,\mathrm{P}}{ab\,(1+\cos\tau)} = b\cos\tau - a\sin\tau, \quad \text{and} \quad \frac{y\,.\,\mathrm{P}}{ab\,(1+\cos\tau)} = b\cos\tau + a\sin\tau;$$

therefore
$$x^2+y^2 = \frac{2a^2b^2\,(1+\cos\tau)^2}{\mathrm{P}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

also
$$(x-y)^2\,\mathrm{P}^2 = 4a^2b^2\,(1+\cos\tau)^2\,a^2\sin^2\tau$$

or
$$(x-y)^2\,\mathrm{P} = 2a^2\sin^2\tau\,(x^2+y^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and
$$(x+y)\;\mathrm{P} = 2ab\,(1+\cos\tau)\,b\cos\tau;$$

from (1),
$$\left\{x^2+y^2-2a\,(x+y)\right\}\mathrm{P} = 2a^2b^2\left\{(1+\cos\tau)^2 - 2\cos\tau\,(1+\cos\tau)\right\}$$
$$= 2a^2b^2\sin^2\tau;$$

from (2),
$$\left\{x^2+y^2-2a\,(x+y)\right\}(x^2+y^2) = b^2\,\frac{2a^2\sin^2\tau\,(x^2+y^2)}{\mathrm{P}} = b^2\,(x-y)^2,$$

or
$$b\left\{a\,(x^2+y^2) + 2d\,(x+y)\right\}(x^2+y^2) = d^2\,(x-y)^2.$$

2. The vector equations to the normals are

$$\rho = \alpha \quad + a\cos\tau \quad + \beta\sin\tau \quad + x\,(b^2\alpha\cos\tau + a^2\beta\sin\tau),$$
$$\rho' = \frac{a}{b}\,\beta + \frac{a}{b}\,\beta\cos\tau - \frac{b}{a}\,\alpha\sin\tau + y\,(ba\beta\cos\tau - ab\alpha\sin\tau);$$

and that these may coincide, we have

$$-abx\,(a^2\sin^2\tau + b^2\cos^2\tau) = a\left\{b\cos\tau - a\sin\tau\right\} + ab + (b^2-a^2)\cos\tau\sin\tau;$$

whence substituting, and putting

$$a^2\sin^2\tau + b^2\cos^2\tau = \mathrm{P} \quad \text{and} \quad a^2\,(1+\cos\tau) - b^2\cos\tau = \mathrm{Q},$$

we get for the required locus

$$ab\mathrm{P}\rho = \sin\tau\,\mathrm{Q}\left\{ba\,(a\sin\tau + b\cos\tau) - \beta a\,(b\cos\tau - a\sin\tau)\right\}.$$

To transform this to a Cartesian equation, we have

$$x\mathrm{P} = \sin\tau\mathrm{Q}\,(a\sin\tau + b\cos\tau), \qquad y\mathrm{P} = \sin\tau\mathrm{Q}\,(a\sin\tau - b\cos\tau);$$

whence we easily find that

$$\frac{b^2\,(x+y)^2 + a^2\,(x-y)^2}{(x^2-y^2)^2}\left\{x^2+y^2-a\,(x+y)\right\}^2$$
$$= \frac{4a^2b^2\sin^2\tau\,\mathrm{Q}^2}{\mathrm{P}^2}\;\frac{\mathrm{P}^4}{a^2b^2\,16\sin^6\tau\,\mathrm{Q}^4\cos^2\tau}\;\frac{4\sin^4\tau\,\mathrm{Q}^2\,(a^2-b^2)^2\cos^2\tau}{\mathrm{P}^2} = (a^2-b^2)^2,$$

or
$$b\left\{a\,(x+y)^2 + b\,(x-y)^2\right\}\left\{a(x^2+y^2) + d(x+y)\right\}^2 = (a-b)^2\,d^2\,(x^2-y^2)^2.$$

3. In the parabola our equations become

$$\mathrm{P}x = t\mathrm{Q}\left\{at-b\right\} \qquad \text{and} \qquad \mathrm{P}y = t\mathrm{Q}\left\{at+b\right\},$$

where P and Q now equal respectively $a^2t^2+b^2$ and $\frac{1}{2}a^2t^2+b^2$.

Let $\tau^2 = b\,t^2 = \frac{1}{4}alt^2$ (where l is the latus rectum); then we have easily

$$xRl = \tau(2\tau - l)\,S \quad \text{and} \quad yRl = \tau(2\tau + l)\,S,$$

where $R = 4\tau^2 + l^2$, and $S = 2\tau^2 + l^2$;

therefore $$l\,\frac{x^2+y^2}{x+y} = \frac{S}{2}, \quad \text{and} \quad -2\tau = l\,\frac{x+y}{x-y};$$

therefore $$4\,\frac{x^2+y^2}{x+y} = l\,\frac{(x+y)^2 + 2(x-y)^2}{(x-y)^2}.$$

II. *Solution by the* PROPOSER; R. TUCKER, M.A.; *and others.*

1. The tangent being $(ax' + d)\,x + by'y + dx' = 0$, the equation to the same line, when the conic is turned through a right angle, will be

$$by'x - (ax' + d)\,y - dx' = 0.$$

Adding these equations, we get $(ax' + d)(x - y) = -by'(x + y)$;

and eliminating y' by means of the equation to the conic,

$$(ax' + d)^2(x-y)^2 + b(ax'^2 + 2dx')(x+y)^2 = 0 \quad \ldots\ldots\ldots (1).$$

Again, eliminating y' between the two equations above, we have

$$(ax' + d)(x^2 + y^2) + dx'(x + y) = 0 \quad \ldots\ldots\ldots\ldots\ldots (2).$$

The elimination of x' between (1) and (2) gives the first locus, which in the case of the parabola evidently reduces to a (circular) cubic.

2. The equations to the normal in the original and new position of the conic are, respectively,

$$by'x - (ax' + d)\,y + y'\left\{(a-b)\,x' + d\right\} = 0,$$

and $$(ax' + d)\,x + by'y + y'\left\{(a-b)\,x' + d\right\} = 0.$$

Pursuing steps similar to those in the case of the tangent, the required locus is easily obtained. In the case of the parabola, developing, reducing, dividing by a, and finally making $a = 0$, there results

$$2(x-y)^2\left\{b(x^2+y^2) + d(x+y)\right\} + d(x+y)^3 = 0.$$

If the given conic be a circle, the locus evidently becomes a circle also.

2839. (Proposed by J. J. WALKER, M.A.)—In a triangle the bisector of the base is equal to the less side and also to one-half of the greater side; determine the three angles.

Solution by the PROPOSER; R. TUCKER, M.A.; *and others.*

Let D be the middle point of the base BC, A the vertex of the triangle, and AC = AD = ½AB. Then AC must bisect the angle between AD and BA produced (Euc. VI. A), and $\angle CAD = 180° - \angle BAC$,

therefore $\angle BAC = 2C$, therefore $3C = 180° - B$,

whence $\sin 3C = 3\sin C - 4\sin^3 C = \sin B$.

Again, $\sin C : \sin B = AB : AC = 2 : 1$, or $\sin C = 2\sin B$,

whence $6\sin C - 8\sin^3 C = \sin C$.

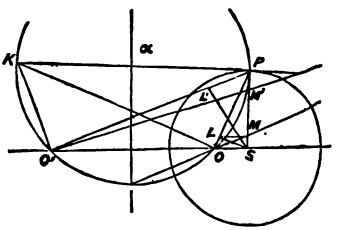
Disregarding the irrelevant solution $C = 0$ or $180°$, we have $\sin^2 C = \frac{5}{8}$.

But $\cos BAC = 1 - 2\sin^2 C = -\frac{1}{4}$, and $\sin^2 B = \frac{1}{4}\sin^2 C = \frac{5}{32}$.

2818. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—Given a circle S and a straight line α not meeting S in real points; O, O′ are the two point-circles to which, and S, α is the radical axis; two conics are drawn osculating S in the same point P, and having one focus at O, O′ respectively; prove that the corresponding directrices coincide.

Solution by R. TUCKER, M.A.

Join OP, O′P, and draw OK, O′K, perpendiculars to them; then it is plain that these lines intersect in K on the circle through O′OP, cutting S orthogonally; and K is the intersection of the directrices. From the centre S draw SL, SL′ perpendiculars on OP, O′P, and LM, L′M′ perpendiculars on SP; then (Salmon's *Conics*, Art. 243) OM, O′M′ are the directions of the axes of the conics, and



$$SM = SP\sin^2 OPS, \quad SM' = SP\sin^2 O'PS;$$

therefore
$$\frac{SM}{SM'} = \frac{\sin^2 OPS}{\sin^2 O'PS} = \frac{OP^2}{O'P^2} = \frac{SO}{SO'};$$

that is, OM is parallel to O′M′, and the directrices (the perpendiculars from K on the axes) coincide.

2761. (Proposed by T. COTTERILL, M.A.)—Find the conic envelope of the radical axis of two circles, one of which is fixed, whilst the other passes through a fixed point. If the radii of the circles are equal, find the foci and axes of the envelope.

Solution by the Rev. J. WOLSTENHOLME, M.A.

Let the fixed point be origin, $x^2 + y^2 - 2\lambda x - 2\mu y = 0$ the equation of the moving circle, and $(x-a)^2 + y^2 = r^2$ that of the fixed circle; then, if r' be the given radius of the moving circle, the radical axis is

$$\lambda x + \mu y - ax + \tfrac{1}{2}(a^2 - r^2) = 0, \text{ with the condition } \lambda^2 + \mu^2 = r'^2;$$

and for the envelope $\frac{x}{\lambda} = \frac{y}{\mu} = \frac{ax - \frac{1}{2}(a^2 - r^2)}{r'^2} = \frac{(x^2+y^2)^{\frac{1}{2}}}{r'}$,

or the envelope is a conic having its focus at the fixed point, its directrix bisecting the tangents drawn from the fixed point to the fixed circle, and eccentricity the ratio of the distance between the fixed point and the centre of the fixed circle to the radius of the moving circle. If the radii be equal, the envelope is the reciprocal polar of the fixed circle with respect to a circle whose centre is the fixed point and radius $\left\{\frac{1}{2}(a^2 - r^2)\right\}^{\frac{1}{2}}$.

2724. (Proposed by S. TEBAY, B.A.)—Three smooth rings, P, Q, R, are thread on an endless string of given length, and constrained to move on three straight rods, OP, OQ, OR. To investigate the motion corresponding to a slight arbitrary disturbance of the system.

Solution by the PROPOSER.

Let $OP = x$, $OQ = y$, $OR = z$,
$PQ = u$, $QR = v$, $RP = w$,

δ_1, δ_2, δ_3 the inclinations of OP, OQ, OR to the vertical; and T the tension of the string;

$POQ = \alpha$, $OPQ = \phi$, $OPR = \phi'$,
$QOR = \beta$, $OQR = \chi$, $OQP = \chi'$,
$ROP = \gamma$, $ORP = \psi$, $ORQ = \psi'$.

Then, for the motions of P, Q, R, we have

$$P\left(\frac{d^2x}{dt^2} - g\cos\delta_1\right) + T(\cos\phi + \cos\phi') = 0 \quad \ldots\ldots\ldots (1),$$

$$Q\left(\frac{d^2y}{dt^2} - g\cos\delta_2\right) + T(\cos\chi + \cos\chi') = 0 \quad \ldots\ldots\ldots (2),$$

$$R\left(\frac{d^2z}{dt^2} - g\cos\delta_3\right) + T(\cos\psi + \cos\psi') = 0 \quad \ldots\ldots\ldots (3).$$

And eliminating T,

$$P(\cos\psi + \cos\psi')\left(\frac{d^2x}{dt^2} - g\cos\delta_1\right) - R(\cos\phi + \cos\phi')\left(\frac{d^2z}{dt^2} - g\cos\delta_3\right) = 0$$
$$\ldots\ldots\ldots (4),$$

$$Q(\cos\psi + \cos\psi')\left(\frac{d^2y}{dt^2} - g\cos\delta_2\right) - R(\cos\chi + \cos\chi')\left(\frac{d^2z}{dt^2} - g\cos\delta_3\right) = 0$$
$$\ldots\ldots\ldots (5).$$

These are the general equations of motion, but their complete solution cannot be effected except for small oscillations.

The geometrical relations give

$u\cos\phi + y\cos\alpha = x$, $v\cos\chi + z\cos\beta = y$, $w\cos\psi + x\cos\gamma = z$,
$w\cos\phi' + z\cos\gamma = x$, $u\cos\chi' + x\cos\alpha = y$, $v\cos\psi' + y\cos\beta = z$.

Let a, b, c be the values of x, y, z, when there is equilibrium; a', b', c' the values of u, v, w; and $\lambda, \lambda', \mu, \mu', \nu, \nu'$ those of $\cos\phi$, $\cos\phi'$, $\cos\chi$, $\cos\chi'$, $\cos\psi$, $\cos\psi'$.

Assume $$x = a + x', \quad y = b + y', \quad z = c + z' \quad \ldots\ldots\ldots\ldots\ldots (6),$$
$$u = a' + u', \quad v = b' + v', \quad w = c' + w' \quad \ldots\ldots\ldots\ldots\ldots (7),$$

where x', y', z', u', v', w' are small quantities.

Now $u^2 = x^2 + y^2 - 2xy\cos\alpha$, $v^2 = y^2 + z^2 - 2yz\cos\beta$, $w^2 = z^2 + x^2 - 2zx\cos\gamma$;

or, neglecting small quantities of orders higher than the first,

$$a'u' = (a - b\cos\alpha)x' + (b - a\cos\alpha)y', \quad b'w' = (b - c\cos\beta)y' + (c - b\cos\beta)z',$$
$$c'w' = (c - a\cos\gamma)z' + (a - c\cos\gamma)x'.$$

But $a - b\cos\alpha = a'$, &c.,

therefore $$u' = \lambda x' + \mu' y', \quad v' = \mu y' + \nu' z', \quad w' = \nu z' + \lambda' x' \quad \ldots\ldots\ldots (8).$$

Hence, from (7),

$$u = a' + \lambda x' + \mu' y', \quad v = b' + \mu y' + \nu' z', \quad w = c' + \lambda' x' + \nu z'.$$

Since $u' + v' + w' = 0$, equations (8) give

$$(\lambda + \lambda')x' + (\mu + \mu')y' + (\nu + \nu')z' = 0;$$

therefore $$(\lambda + \lambda')\frac{d^2x'}{dt^2} + (\mu + \mu')\frac{d^2y'}{dt^2} + (\nu + \nu')\frac{d^2z'}{dt^2} = 0.$$

Also, from (6), $$\frac{d^2x}{dt^2} = \frac{d^2x'}{dt^2}, \quad \frac{d^2y}{dt^2} = \frac{d^2y'}{dt^2}, \quad \frac{d^2z}{dt^2} = \frac{d^2z'}{dt^2}.$$

Substituting these in (4) and (5), neglecting small quantities of orders higher than the first, and putting

$$A = \frac{\lambda}{a'}(\nu + \nu') + \frac{1}{c'}(\lambda'\nu' - \cos\gamma) - \left(\frac{1}{b'} + \frac{1}{c'}\right)(1 + \nu\nu')\frac{\lambda + \lambda'}{\nu + \nu'},$$

$$B = \left\{\frac{\nu'}{b'}(\lambda + \lambda') + \frac{1}{c'}(\lambda\nu - \cos\gamma)\right\}\frac{\lambda + \lambda'}{\nu + \nu'} - \left(\frac{1}{a'} + \frac{1}{c'}\right)(1 + \lambda\lambda'),$$

$$C = \frac{\mu'}{a'}(\nu + \nu') + \frac{1}{b'}(\mu\nu - \cos\beta) - \left(\frac{1}{b'} + \frac{1}{c'}\right)(1 + \nu\nu')\frac{\mu + \mu'}{\nu + \nu'},$$

$$D = \left\{\frac{\nu'}{b'}(\lambda + \lambda') + \frac{1}{a'}(\lambda\nu - \cos\gamma)\right\}\frac{\mu + \mu'}{\nu + \nu'} - \frac{\mu}{b'}(\lambda + \lambda') - \frac{1}{a'}(\lambda'\mu' - \cos\alpha),$$

$$A' = \frac{\lambda}{a'}(\nu + \nu') + \frac{1}{c'}(\lambda\nu' - \cos\gamma) - \left(\frac{1}{b'} + \frac{1}{c'}\right)(1 + \nu\nu')\frac{\lambda + \lambda'}{\nu + \nu'},$$

$$B' = \left\{\frac{\nu}{c'}(\mu + \mu') + \frac{1}{b'}(\mu'\nu' - \cos\beta)\right\}\frac{\lambda + \lambda'}{\nu + \nu'} - \frac{\lambda'}{c'}(\mu + \mu') - \frac{1}{a'}(\lambda\mu - \cos\alpha),$$

$$C' = \frac{\mu'}{a'}(\nu + \nu') + \frac{1}{b'}(\mu\nu' - \cos\gamma) - \left(\frac{1}{b'} + \frac{1}{c'}\right)(1 + \nu\nu')\frac{\mu + \mu'}{\nu + \nu'},$$

$$D = \left\{\frac{\nu}{a'}(\mu + \mu') + \frac{1}{b'}(\mu'\nu' - \cos\beta)\right\}\frac{\mu + \mu'}{\nu + \nu'} - \left(\frac{1}{a'} + \frac{1}{b'}\right)(1 + \mu\mu'),$$

we find, after reduction,

$$\frac{P(\nu + \nu')^2 + Q(\lambda + \lambda')^2}{\nu + \nu'} \cdot \frac{d^2x'}{dt^2} + \frac{R(\lambda + \lambda')(\mu + \mu')}{\nu + \nu'} \cdot \frac{d^2y'}{dt^2}$$
$$= g(PA\cos\delta_1 + RB\cos\delta_2) + g(PC\cos\delta_1 + RD\cos\delta_2),$$

$$\frac{R(\lambda+\lambda')(\mu+\mu')}{\nu+\nu'}\cdot\frac{d^2x'}{dt^2}+\frac{Q(\nu+\nu')^2+R(\mu+\mu')^2}{\nu+\nu'}\cdot\frac{d^2y'}{dt^2}$$
$$=g(QA'\cos\delta_2+RB'\cos\delta_3)+g(QC'\cos\delta_2+RD'\cos\delta_3).$$

Again, putting

$$h=QR(\lambda+\lambda')^2+RP(\mu+\mu')^2+PQ(\nu+\nu')^2,$$

$$k=\left\{Q(\nu+\nu')^2+R(\mu+\mu')^2\right\}(PA\cos\delta_1+RB\cos\delta_3)$$
$$-R(\lambda+\lambda')(\mu+\mu')(QA'\cos\delta_2+RB'\cos\delta_3),$$

$$l=\left\{R(\mu+\mu')^2+Q(\nu+\nu')^2\right\}(PC\cos\delta_1+RD\cos\delta_3)$$
$$-R(\lambda+\lambda')(\mu+\mu')(QC'\cos\delta_2+RD'\cos\delta_3),$$

$$k'=\left\{R(\lambda+\lambda')^2+P(\nu+\nu')^2\right\}(QA'\cos\delta_2+RB'\cos\delta_3)$$
$$-R(\lambda+\lambda')(\mu+\mu')(PA\cos\delta_1+RB\cos\delta_3),$$

$$l'=\left\{P(\nu+\nu')^2+R(\lambda+\lambda')^2\right\}(QC'\cos\delta_2+RD'\cos\delta_3)$$
$$-R(\lambda+\lambda')(\mu+\mu')(PC\cos\delta_1+RD\cos\delta_3),$$

we finally get
$$h(\nu+\nu)\frac{d^2x'}{dt^2}-g(kx'+ly')=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(9),$$
$$h(\nu+\nu')\frac{d^2y'}{dt^2}-g(k'x'+l'y')=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

Hence, for small oscillations, the motion of the system is given by the simultaneous equations (9), (10), and will consist of two independent vibrations, of which the periods are

$$\pi\left\{\frac{h(\nu+\nu')}{g(k+k'\eta')}\right\}^{\frac{1}{2}},\qquad \pi\left\{\frac{h(\nu+\nu')}{g(k+k'\eta'')}\right\}^{\frac{1}{2}};$$

where η', η'' are the roots of the equation $k'\eta^2+(k-l')\eta=l$.

2642. (Proposed by C. W. MERRIFIELD, F.R.S.)—If three surfaces of the second degree meet in four points, at each of which their three curves of intersection have a common tangent, these four points lie in one plane.

Solution by the PROPOSER.

Write the equation of the quadric in its general form

$$F=ax^2+by^2+cz^2+d^2+2lyz+2mzx+2nxy+2px+2qy+2rz=0,$$

and use suffixes to distinguish the three surfaces F_1, F_2, F_3.

Taking our origin at one of the points, and making the axes of x and y pass through two others $(2h, 0, 0)$ and $(0, 2k, 0)$, we get $d=0$ and $p=-ah$, $q=-bk$, where h and k are independent of the suffix. Take, moreover, the common tangent at the origin for the axis of z, and we have r_1, r_2, r_3 each $=0$.

The tangent planes at the origin may therefore be written as

$$a_1hx + b_1ky = 0, \quad a_2hx + b_2ky = 0, \quad a_3hx + b_3ky = 0;$$

and as these are to have a common intersection, we get

$$\frac{a_1-a_2}{a_3} = \frac{b_1-b_2}{b_3}.$$

Treating in a similar manner the points $(2h, 0, 0)$ and $(0, 2k, 0)$, for which the tangent planes are

$$(x-2h)\,ah + y\,(2nh-bk) + z\,.\,2mh = 0$$

and

$$x\,(2nk-ah) + (y-2k)\,bk + z\,.\,2lk = 0,$$

we get the further relations

$$\frac{a_1-a_2}{a_3} = \frac{m_1-m_2}{m_3} = \frac{2\,(n_1-n_2)\,h-(b_1-b_2)\,k}{2n_3h-b_3k},$$

$$\frac{b_1-b_2}{b_3} = \frac{l_1-l_2}{l_3} = \frac{2\,(n_1-n_2)\,k-(a_1-a_2)\,h}{2n_3k-a_3h},$$

and since $\frac{a_1-a_2}{a_3} = \frac{b_1-b_2}{b_3}$, these are all equal, and so is also $\frac{n_1-n_2}{n_3}$. We thus get the following reductions in the general equation:—

$$d = 0, \quad r = 0, \quad \frac{p_1}{a_1} = \frac{p_2}{a_2} = \frac{p_3}{a_3} = -h, \quad \frac{q_1}{a_1} = \frac{q_2}{a_2} = \frac{q_3}{a_3} = -k,$$

$$\frac{a_1-a_2}{a_3} = \frac{b_1-b_2}{b_3} = \frac{l_1-l_2}{l_3} = \frac{m_1-m_2}{m_3} = \frac{n_1-n_2}{n_3} (= g);$$

and we may therefore write the three quadrics as

$$F_1 = a_1x^2 + b_1y^2 + c_1z^2 + 2l_1yz + 2m_1zx + 2n_1xy - 2a_1hx - 2b_1ky = 0,$$

$$F_2 = a_2x^2 + \ldots\ldots\ldots$$

$$F_3 = \frac{a_1-a_2}{g}x^2 + \frac{b_1-b_2}{g}y^2 + c_3z^2 + 2\frac{l_1-l_2}{g}yz + \ldots\ldots\ldots = 0,$$

the only outlying term being c_3.

But $$F_3 - \frac{F_1-F_2}{g} = 0 \text{ gives } \frac{gc_3-c_1+c_2}{g}z^2 = 0,$$

or there is a quadric through the intersection which degenerates into the double plane $z=0$. This proves the theorem, which is easily reciprocated. The investigation, moreover, shows that, if there be three such points, there will also be a fourth.

2771. (From WOLSTENHOLME'S *Book of Mathematical Problems.*)—Prove that (1) if a straight line of length a be divided at random in two points, the mean value of the sum of the squares on the three parts is $\frac{1}{2}a^2$; (2) if the line be divided at random into two parts, and the longer part again divided at random into two parts, the mean value of the sum of the squares

on the three parts is $\frac{3}{7}\frac{5}{2}a^2$; and (3) if it is an even chance that n times the sum of the squares on the parts in (1) is less than the square on the whole line, then $n = \dfrac{12\pi}{4\pi + 3\sqrt{3}}$.

I. *Solution by* STEPHEN WATSON.

(1.) Let x, y, and $a-x-y$ be the parts. Then the total of cases is a^2, and doubling because of the interchange of x and y, the average is

$$\frac{2}{a^2}\int_0^a dx \int_0^{a-x} \left\{x^2+y^2+(a-x-y)^2\right\} dy = \tfrac{1}{4}a^2.$$

(2.) Here the number of cases is

$$2\int_0^{\frac{1}{2}a} (a-x)\,dx = \tfrac{3}{4}a^2,$$

and the average is

$$2\int_0^{\frac{1}{2}a} dx \int_0^{a-x} \left\{x^2+y^2+(a-x-y)^2\right\} dy + \tfrac{3}{4}a^2 = \tfrac{3}{7}\tfrac{5}{2}a^2.$$

(3.) In this case, let x, $\frac{1}{2}(a-x)-y$, $\frac{1}{2}(a-x)$ be the parts; then

$$x^2+\tfrac{1}{2}(a-x)^2+2y^2 < \frac{a^2}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

therefore

$$y < \pm\tfrac{1}{2}(ma^2-3z^2)^{\frac{1}{2}},$$

where $m = \dfrac{2}{n} - \dfrac{2}{3}$ and $z = x-\frac{1}{2}a$. Hence the number of positions y can take for each value of x is $(ma^2-3z^2)^{\frac{1}{2}}$, and therefore the chance p of (1) being fulfilled is, since $dx=dz$,

$$p = \frac{2}{a^2}\int_{-(\frac{1}{3}m)^{\frac{1}{2}}a}^{(\frac{1}{3}m)^{\frac{1}{2}}a} (ma^2-3z^2)^{\frac{1}{2}}dz = \frac{m\pi}{\sqrt{3}} = \left(\frac{2}{n}-\frac{2}{3}\right)\frac{\pi}{\sqrt{3}}.$$

therefore

$$n = \frac{6\pi}{2\pi+3p\sqrt{3}}, \quad (\text{when } p=\tfrac{1}{2}) = \frac{12\pi}{4\pi+3\sqrt{3}}.$$

II. *Solution by the* REV. J. WOLSTENHOLME, M.A.

(1.) The answer to this part of the question is

$$\int_0^a \int_x^a \left\{x^2+(y-x)^2+(a-y)^2\right\} dx\,dy \div \int_0^a \int_x^a dx\,dy$$

$$= 2a^2\int_0^1 \left\{(1+2x^2)(1-x)-(1+x)^2(1-x^2)+\tfrac{2}{3}(1-x^3)\right\} dx$$

$$= \tfrac{1}{4}a^2\int_0^1 (2-6x+9x^2-5x^3) = 2a^2(2-3+3-\tfrac{5}{4}) = \tfrac{1}{2}a^2.$$

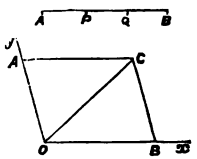
(2.) The answer to this part of the question is

$$\int_0^{\frac{a}{2}}\int_0^{a-x}\left\{x^2+y^2+(a-x-y)^2\right\}dx\,dy \div \int_0^{\frac{a}{2}}\int_0^{a-x}dx\,dy$$

$$= \frac{8a^2}{3}\int_0^{\frac{1}{2}}\left\{x^2+\tfrac{2}{3}(1-x)^2\right\}(1-x)\,dx = \frac{8a^2}{9}\int_0^{\frac{1}{2}}(2-6x+9x^2-5x^3)\,dx$$

$$= \frac{8a^2}{9}\left(1-\frac{6}{8}+\frac{9}{24}-\frac{5}{64}\right) = \frac{a^2}{72}(64-48+24-5) = \frac{35}{72}a^2.$$

(3.) If AB be the rod; P, Q the points of division, P nearer A; AP $= x$, AQ $= Y$; then let us find the chance that $x^2+(y-x)^2+(a-y)^2 < \frac{a^2}{n}$,



where $x>0<a$, $y>x<a$.

Take axes Ox, Oy inclined at 120°; measure OA $=$ OB $= a$, and complete the rhombus OACB; then to every possible division of the rod corresponds a point in the triangle OAC, and to every favourable division a point within the circle

$$x^2+y^2-xy-ay+\frac{a^2}{2} = \frac{a^2}{2n};$$

or, transferring to the centre $X^2-XY+Y^2 = \frac{a^2}{2n}-\frac{a^2}{6}$.

When $n=2$, this is the inscribed circle; and when $n>2$, the circle lies altogether within the triangle, and

$$\text{its area} : \text{area of inscribed circle} = \frac{1}{n}-\frac{1}{3} : \frac{1}{6} = 2(3-n) : n,$$

and area of inscribed circle : area of triangle $= \pi : 3\sqrt{3}$;

therefore chance we are seeking is $\frac{2\pi(3-n)}{3n\sqrt{3}} = \frac{1}{2}$, if $n = \frac{12\pi}{4\pi+3\sqrt{3}}$;

and since this is greater than 2, it is the value required. If $n<2$, the chance that n times sum of squares is less than the square on whole is

$$\frac{3-n}{3n\sqrt{3}}(3\sin 2\alpha + 2\pi - 6\alpha), \quad \text{where } \sin\alpha = \left(\frac{6-3n}{6-2n}\right)^{\frac{1}{2}}.$$

The chance that 3 times squares on parts $>$ 2 square on whole is $\frac{\pi+6}{6\sqrt{3}}$,

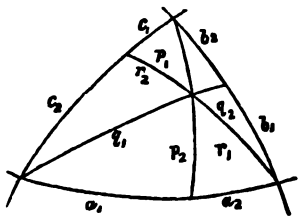
" 7 " " $>$ 5 square on whole is $\frac{1}{3}+\frac{2\pi}{9\sqrt{3}}$.

2455. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—If a, b, c be the three sides of a spherical triangle, and k the radius of its polar circle, prove the formula

$$\tan^2 k = \frac{(\cos b\cos c\sec a-1)(\cos c\cos a\sec b-1)(\cos a\cos b\sec c-1)}{4\sin s\sin(s-a)\sin(s-b)\sin(s-c)}.$$

Solution by the PROPOSER.

If p, q, r be the three perpendiculars of the triangle; a_1 and a_2, b_1 and b_2, c_1 and c_2 the three pairs of segments into which they divide the three sides; A_1 and A_2, B_1 and B_2, C_1 and C_2 the three pairs into which they divide the three angles; p_1 and p_2, q_1 and q_2, r_1 and r_2 the three pairs into which they divide each other; and k as above; then, since



$$\tan p_1 \tan p_2 = \tan q_1 \tan q_2 = \tan r_1 \tan r_2 = -\tan^2 k,$$

and since, by Napier's rule,

$$\tan p \tan p_1 = \tan c_1 \tan c_2 = \tan b \tan c \cos A$$
$$\tan p_2 = \sin a_1 \tan B_1 = \tan p \cot B \cot C \sec a,$$

therefore, at once, $\tan^2 k = -\dfrac{\sin b \sin c}{\sin B \sin C} \cdot \dfrac{\cos A \cos B \cos C}{\cos a \cos b \cos c}$,

which, substituting for the functions of the angles their familiar values in terms of the sides, gives immediately the above.

A process exactly similar leads to the corresponding formula for a plane triangle; viz.,

$$k^2 = -\frac{\sin b \sin c}{\sin B \sin C} \cos A \cos B \cos C = -\frac{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}{32s(s-a)(s-b)(s-c)}$$

[See Townsend's *Modern Geometry*, Vol. I., art. 168.]

2877. (Proposed by Professor SYLVESTER.)—If

$$u = x+y+z+t, \qquad v = fx+gy+hz+kt,$$
$$U = yzt+ztx+txy+xyz, \qquad V = fyzt+gztx+htxy+kxyz;$$

prove that the resultant of u, v, U, V is

$$(f-g)(f-h)(f-k)(g-h)(g-k)(h-k)(f+g-h-k)^2(f+h-g-k)^2 \times (f+k-g-h)^2.$$

Solution by the PROPOSER.

Suppose $x+y=0$, $z=0$, $t=0$; then all four equations are satisfied provided $f=g=0$. Hence $(f-g)(f-h)(g-h)(f-k)(g-k)(h-k)$ is a factor of the resultant.

Again, suppose $x=y=-z=-t$; all four equations are satisfied provided $f+g=h+k$. Hence $(f+g-h-k)(f+h-g-k)(g+h-f-k)$ is also a factor of the same.

But furthermore, if $f+g-h-k=0$, then at the point of intersection $x=1$, $y=1$, $z=-1$, $t=-1$, it may easily be verified that the planes u, U

respectively touch the cubic surfaces v, V. Either of these facts warrants us in concluding that some power of $(f+g-h-k)$ *higher than the first* must enter into the resultant. Hence

$$(f-g)(f-h)(f-k)(g-k)(g-k)(h-k)$$
$$\times \left\{(f+g-h-k)(f+h-g-k)(g+h-f-k)\right\}^2$$

is contained in the resultant. But the order of the resultant in these letters is evidently 3^2+3, *i.e.* 12. Hence the above quantity is the complete resultant.

I obtained the theorem originally by ordinary Algebra, I forget how, probably by a direct application of the dialytic process; but the preceding method is more instructive, as embodying in a simple instance the transcendental law familiar to eliminationists, that *any singularity of relation between loci intersecting in virtue of a condition* implies the appearance of a power, higher than the first, of the characteristic of that condition in the complete resultant. The theorem was wanted in an enquiry connected with the subject of rectifiable compound logarithmic waves. A vast extension of it will be given in the answer to a subsequent question.

Note on the late Judge Hargreave's Solution of the Quintic.

By the Rev. T. P. Kirkman, M.A., F.R.S.

One of the final equations at page 94 of the Judge's posthumous treatise is

$$y_1^{\frac{1}{5}} + y_2^{\frac{1}{5}} = A_1(z_1-b_1) + A_2(z_2-b_1) + A_3(z_3-b_1) + A_4(z_4-b_1) + A_5(z_5-b_1)$$
$$= \Sigma(At).$$

If so, it is possible to find $\alpha_1+\beta_1 = A_1$, $\alpha_2+\beta_2 = A_2$, &c., such that

$$y_1^{\frac{1}{5}} + y_2^{\frac{1}{5}} = \left\{(\Sigma\alpha t)^5\right\}^{\frac{1}{5}} + \left\{(\Sigma\beta t)^5\right\}^{\frac{1}{5}} = (P^5)^{\frac{1}{5}} + (Q^5)^{\frac{1}{5}},$$

where $P^5 = y_1$, and $Q^5 = y_2$.

As $y_1+y_2 = 2(51)$, p. 85, is a rational and symmetrical function of the z's, P^5+Q^5 is one also; that is, P^5 is a two-valued function, whose values are P^5 and Q^5. It follows that P is also a two-valued function of the z's, whose two values are P and Q; for if P had more than these two values, PPPPP+QQQQQ could not be a rational and symmetrical function. Hence $P+Q = \Sigma(At)$ is a symmetrical function of the z's. That is,

$$y_1^{\frac{1}{5}} + y_2^{\frac{1}{5}} = \Sigma(At) = A\Sigma t = A\Sigma(z-b_1) = 0,$$

and $$y_1 = -y_2, \quad \text{or} \quad y_1+y_2 = 2(51) = 0.$$

This destroys the form of the conditioned quintic in z, at p. 82; for $(51)=0$ gives, as (21) and (31) are each $= 0$, (p. 78), (p. 7),

$$(51) = 4b_1^5 - 5b_1b_4 + b_5 = 0,$$

or b_5 is expressible in b_1 and b_4. Thus it seems that what Hargreave has been solving is not the general quintic.

2870. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—Given three points A, B, C, and a conic; two points P, Q are taken on the conic such that the pencil $A\{BPQC\}$ is harmonic; prove that the envelope of PQ is a conic touching AB, AC at points on the polar of A with respect to the given conic.

Solution by ARCHER STANLEY.

Since the intersections x and x' of every connector PQ and the given lines AB, AC are harmonic conjugates with respect to P and Q, the polar of one must pass through the other; in other words, x and x' are conjugate points relative to the conic. Conversely, every connector of a pair of conjugate points x, x' on AB and AC, being harmonically divided by its intersections P, Q with the conic, must be a tangent of the required envelope. But to a point x on AB there is but one conjugate x' on AC, and *vice versâ*; hence x and x' are corresponding points of two homographic rows, and their connector xx' envelopes a conic which touches AB, AC at the points which correspond to A, that is to say, at the points B, C, where AB and AC are intersected by the polar of A. This envelope has obviously, in common with the given conic, the four tangents at the points where AB and AC cut that conic.

The following are amongst the most interesting special cases:

1. If ABC be a self-conjugate triangle, the envelope degenerates to the *point-pair* (B, C).

2. The envelope also degenerates to a point-pair (A, A'), if A be on the given conic, and A' be the intersection of the tangents at the points where AB and AC cut the conic again.

3. If AB and AC touch the conic in B and C, the envelope will likewise touch it in these points.

4. If A be at the centre of the given conic, the envelope will be a concentric conic having AB, AC for its asymptotes. If AB, AC be themselves asymptotes of the given conic, then the envelope will be a concentric, similar, and similarly placed conic.

Many well known theorems in conics immediately follow from the above special cases, wherein, it should be observed, AB and AC may be imaginary. For instance, if these lines pass through the circular points at infinity, then the envelope under consideration will become that of chords of the given conic which subtend a right angle at A, and this envelope will have A for its focus. If A be itself a focus of the given conic, then the envelope (by 3) will have in common with it this focus and the corresponding directrix. If A be on the conic, however, then (by 2) the chords which subtend a right angle at A will all pass through a point A' on the normal at A. (Salmon's *Conics*, Art. 181, Ex. 2.)

2828. (Proposed by S. TEBAY, B.A.)—A straight rod is divided into n parts in arithmetical progression, and equal particles are fixed at the points of division. If the system be made to vibrate about one extremity, determine the average length of the pendulum, neglecting the weight of the rod.

Solution by the PROPOSER.

Let a be the length of the rod, L the length of the pendulum for any adjustment of the particles; $u_1, u_2, \ldots u_{n-1}$ their distances from the point of suspension. Then

$$L = \frac{u_1^2 + u_2^2 + \ldots + u_{n-1}^2}{u_1 + u_2 + \ldots + u_{n-1}} = \frac{\Sigma(u_n^2)}{\Sigma(u_n)}.$$

Let $u_1 = x$, $x_2 - u_1 = x + y$, where y is the common difference.

Therefore $$u_n = \tfrac{1}{2}n\{2x + (n-1)y\};$$

$$\Sigma(u_n^2) = \tfrac{1}{12}n(n-1)(2n-1)(2x-y)^2 + \tfrac{1}{8}n^2(n-1)^2(2x-y)y + \tfrac{1}{120}n(n-1)(2n-1)(3n^2-3n-1)y^2,$$

$$\Sigma(u_n) = \tfrac{1}{2}n(n-1)(2x-y) + \tfrac{1}{12}n(n-1)(2n-1)y.$$

But $\tfrac{1}{2}n\{2x + (n-1)y\} = a$; therefore $y = \dfrac{2(a-nx)}{n(n-1)}$;

therefore $$\Sigma(u_n^2) = \frac{1}{30n(n-1)}\left\{\frac{n^4-1}{n-1}n^2x^2 + an(n-2)(n+1)(3n+1)x + 2a^2(n-2)(3n^2-6n+1)\right\},$$

$$\Sigma(u_n) = \tfrac{1}{6}n(n+1)x + \tfrac{1}{3}a(n-2).$$

Now the number of pendulums is proportional to $\dfrac{a}{n}$, this being the greatest value of x. Therefore the average length of the pendulum is

$$\frac{n}{a}\int_0^{\frac{a}{n}} \frac{\Sigma(u_n^2)}{\Sigma(u_n)} = \frac{a}{10n(n^2-1)}\left\{3n^3 + 5n^2 - 17n + 5 + 4\frac{(n-1)^2(n-2)(2n-1)}{n+1}\log\frac{3(n-1)}{2(n-2)}\right\}.$$

If n be increased indefinitely, this becomes

$$\frac{a}{5}\left(\frac{3}{2} + 4\log\frac{3}{2}\right) = .6243\,a.$$

If the distances of the particles from the point of suspension be in arithmetical progression, and one particle be attached to the lower extremity of the rod, the average length of the pendulum is

$$\frac{a}{3(n-1)}\left\{\tfrac{1}{2}(2n-7) + 2(n+1)\log 2\right\}.$$

2533. (Proposed by Professor HIRST.) — Find the envelope of a line upon which two given conics intercept segments which have a common middle point; and find also the locus of this middle point.

I. *Solution by the* Rev. J. WOLSTENHOLME, M.A.

If the two conics be $lx^2+my^2+nz^2=0$, $l'x^2+m'y^2+n'z^2=0$, and $ux+vy+wz=0$ be a straight line divided as required, we must have

$$(mn'-m'n)\frac{u}{v-w}+(nl'-n'l)\frac{v}{w-u}+(lm'-l'm)\frac{w}{u-v}=0,$$

the tangential equation of the envelope. For the locus of the middle point, we have "The envelope of a chord of a conic bisected by a given straight line is a parabola of which that straight line is a tangent." Hence the points in which a given straight line will meet the locus are the points in which it will be met by the two remaining common tangents to two parabolas; two points: or the locus is a conic.

II. *Solution by the* PROPOSER.

Let (C) and (C′) be the two given conics, L one of the lines whose envelope is required, and l the common middle point of the two segments which (C) and (C′) determine upon L. It is obvious that the intersections of (C) and (C′) with L determine an involution of which l is a double point, the other being at infinity; and it is well known that every conic passing through the intersections of (C) and (C′) will cut L in a pair of conjugate points of this involution, that is to say, will intercept on L a segment which has the same point l for its middle point. One of the conics of this pencil, in fact, will touch L at l, another will have L for its asymptote, and consequently have its centre on L. Now the locus of the centres of conics of the pencil being a conic (Σ), there will be another conic which has likewise its centre on L, and that centre must obviously be the common middle point l.

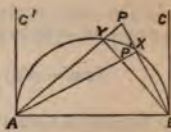
Conversely, every point on (Σ) is a point l on some line L. For, besides being the centre of a conic of the pencil, it is the middle point of one chord of (C). Two conics of the pencil, therefore, intercept segments on this chord which have a common middle point l; hence all do so, and the chord is a position of L. The required locus of l, therefore, is the conic of centres Σ.

Moreover, the above investigation has shown that the required envelope is that of the asymptotes of the conics of the pencil $\{(C), (C')\}$. Hence its class can be easily shown to be 3. For through an arbitrary point x at infinity only one conic of the pencil passes, and therefore only one proper asymptote. The pencil however contains two parabolas, so that the line at infinity must be counted twice as an asymptote through x.

The required envelope being of the third class, and having a double tangent at infinity must, of course, be of the fourth order.

2835. (Proposed by T. COTTERILL, M.A.)—In a plane, if A, B are two points, and a point P describe a curve of the nth order, show that P′, the intersection of the perpendiculars in the triangle PAB, will describe a curve of the $2n$th order with three multiple points of the order n. Explain why the curve corresponding to a circle through the points A and B includes its reflexion to the line AB.

Solution by the PROPOSER.



(1.) Let AC, BC be the perpendiculars to the line AB at the points A, B. Then, if P be a point in the plane, not on the lines, and PA, PB be joined, the perpendiculars AX, BY on PB, PA will intersect in a point P′; the pair will conjugately correspond, since each point is derived from the other by the same construction. The pair lie on a line through C, which is made up of such pairs of points; amongst the lines is the line at infinity.

Let A, B, C be called principal points. Then, if P lie on the line AC, P′ coincides with A. Similarly, to a non-principal point on BC the point B always corresponds. If P is a non-principal point on AB, the point C corresponds. Thus to every non-principal point of the plane one and but one point corresponds, whilst to each of the principal points the points of a line or the line itself corresponds. To a curve, therefore, of the nth order, which does not pass through a principal point, a proper curve of the $2n$th order will correspond, having multiple points of the order n at the points A, B, C. The nature of the correspondence of such curves is more clearly seen by observing (2) that since the angles at X and Y are right angles, the points X and Y lie on the circle described on AB as a diameter, and that P, P′ are harmonic conjugates to the circle. But the line PP′ always passes through C, the pole of the diameter AB; so that we have a particular case of quadric inversion, the general principles of which have been fully explained by Prof. HIRST in his Paper on Quadric Inversion contained in the *Proceedings of the Royal Society* for March, 1865. Corresponding points are therefore inverse points. One peculiarity of this inversion is that a right hyperbola through A, B is its own inverse; and since the pole of PP′ to such a conic passing through PP′ is on XY, it follows that tangents to a curve at P and its inverse at P′ meet on XY.

(3.) The algebraical equations of the correspondence are very simple. Let (u, v, y), (u', v', y') be the perpendiculars from PP′ on the lines AC, BC, and AB, the line PP′ cutting AB in M. Then $PM \cdot P'M = AM \cdot MB$, so that we have the equations $u' = u$, $v' = v$, and $y'y = uv$. Also

$$y' + \frac{u'v'}{y'} = y + \frac{uv}{y}, \quad y' - \frac{u'v'}{y'} = -\left(y - \frac{uv}{y}\right) \text{ &c.}$$

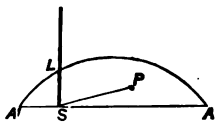
The equation to the circle AXB, the locus of coincident inverse points, is $uv - y^2 = K = 0$. The right hyperbola (axis AB) is $uv + y^2 = H = 0$. If $(u, v, y) = 0$ be the homogeneous equation to a curve, (uy, vy, uv) is its inverse; $(uy, vy, H) = 0$ is its own inverse; the inverse of $(uy, vy, K) = 0$ is its reflexion to the line AB, &c.

In tracing a curve which has a simple inverse, the (xy) system of coordinates is often convenient, in which x is the perpendicular from P on OC, O being the centre of AB, y remaining the same as before. If $AB = 2a$, the inverse coordinates are connected by the equations $x' = x$, $y'y = a^2 - x^2$. Thus the inverse of the circle $(y + b)^2 = a^2 - x^2$, which touches the lines AC, BC, is $(by + a^2 - x^2)^2 = (a^2 - x^2)\, y^2$, a quartic curve having cusps at A and B, and a double point at C. If the circle does not cut the line AB, the quartic, by its revolution round OC, will generate a surface in the shape of a bell with its mouth uppermost, containing a bowl which just fits it at the rim.

The inverse of a circle through AB is its reflexion to the line AB, because any point and the reflexion of its inverse to the line AB are on a circle through AB, as is easily seen.

2821. (From WOLSTENHOLME'S *Book of Mathematical Problems.*)—Show that the mean value of the distance from one of the foci of all points within a given prolate spheroid is $\frac{1}{4}a(3+e^2)$, $2a$ being the axis and e the eccentricity.

Solution by R. TUCKER, M.A.



Let P be an element of the ellipse at a distance r from the focus (S), and the angle $ASP = \theta$; then, by the revolution of the curve about AS, the element at P generates a small ring equal to $2\pi r \sin\theta\, r\,\Delta\theta\,\Delta r$. The volume (V) generated by $ASL = \frac{1}{2}\pi a^3(1-e^2)(1+e)^2(2-e)$, and the volume (V′) generated by $A'SL = \frac{1}{2}\pi a^3(1-e^2)(1-e)^2(2+e)$;

hence $\dfrac{V}{V+V'} = \dfrac{(2-e)(1+e)^2}{4}$, and $\dfrac{V'}{V+V'} = \dfrac{(2+e)(1-e)^2}{4}$.

For the part to the right of the bounding plane through SL, putting $r' = \dfrac{a(1-e^2)}{1-e\cos\theta}$, the average distance of P is

$$\frac{2\pi}{V}\int_0^{\frac{1}{2}\pi}\int_0^{r'} r^3 \sin\theta\, d\theta\, dr = \frac{a}{2}\left(\frac{1+e}{2-e}\right)(3-3e+e^2) \quad \ldots\ldots\ldots (\alpha).$$

The average distance for the portion to the left is found by changing e into $-e$; thus it equals

$$\frac{a}{2}\left(\frac{1-e}{2+e}\right)(3+3e+e^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots (\beta).$$

The average distance required must be determined proportionally to the volumes; thus it equals

$$\frac{\alpha V+\beta V'}{V+V'} = \frac{a}{8}\left\{\frac{(1+e)^3}{e}\left\{1-(1-e)^3\right\} + \frac{(1-e)^3}{e}\left\{(1+e)^3-1\right\}\right\} = \frac{(3+e^2)a}{4}.$$

2815. (Proposed by A. MARTIN.)—A cask contains a gallons of wine. Through a hole in the top water or wine can be let in at the rate of b gallons per minute; and through a pipe in the bottom, when open, the mixture can escape at the same rate. Suppose the discharge pipe is opened at the same instant that water is let in at the top, and t minutes afterwards the water is shut off and wine let in. Required the quantity of water in the cask at the end of t_1 minutes from the opening of the discharge pipe, and the length of time elapsed, both before and after wine was let in at the top, when the quantities of the two fluids in the cask were equal, supposing them to mingle perfectly.

Solution by C. R. RIPPIN, M.A.; R. J. NELSON, M.A.; *the* PROPOSER; *and others.*

Suppose both pipes to have been open for any time t; and let t be divided into n equal intervals, and conceive the alteration in the fluid to have been

effected by removing $\frac{bt}{n}$ gallons in each interval and replenishing it with $\frac{bt}{n}$ gallons of water.

Then, if V′ be the number of gals. of wine at the beginning of an interval, and V″ be the number of gals. of wine at the end of the same interval,

we have $V'' = V' - \frac{V'}{a} \cdot \frac{bt}{n} = V'\left(1 - \frac{bt}{an}\right) = V'c$ (c being constant);

so that, if V be the quantity of wine at end of time t, then, since there were a gallons at first, we have

$$V = ac^n = a\left(1 - \frac{bt}{an}\right)^n.$$

Hence, making n infinite, we have for the result in the proposed question, that the wine at end of time $t = a\epsilon^{-\frac{bt}{a}}$, and therefore the water $= a(1-\epsilon^{-\frac{bt}{a}})$.

Denoting this by a', we have, by the same process, that the water in the cask after a further time $t' = a'\epsilon^{-\frac{bt'}{a}} = a'\epsilon^{-\frac{b}{a}(t_1 - t)}$; and the wine and water

will be equal when $a\epsilon^{-\frac{bt}{a}} = \frac{a}{2}$ or $t = \frac{a}{b}\log 2$ .....(1),

and also when $a'\epsilon^{-\frac{bt'}{a}} = \frac{a}{2}$ or $t' = \frac{a}{b}\log\frac{2a'}{a}$ .....(2).

2816. (Proposed by the EDITOR.)—Find three square numbers in arithmetical progression, such that the square root of each (α) increased or else (β) diminished by unity shall give three rational squares.

I. *Solution by* SAMUEL BILLS.

If we put $a = 2pq + (p^2 - q^2)$, $b = 2pq - (p^2 - q^2)$, and $c = p^2 + q^2$, then will a^2, b^2, c^2 be three square numbers in arithmetical progression.

Now put $\frac{ax}{y^2}, \frac{bx}{y^2}, \frac{cx}{y^2}$ for the roots of the three squares required in the question; then in the part (α) of the question we shall have to find

$$\frac{ax}{y^2} + 1, \frac{bx}{y^2} + 1, \frac{cx}{y^2} + 1, \text{ or } ax + y^2,\ bx + y^2,\ cx + y^2 \text{ all squares.}$$

Assume $ax + y^2 = (r+y)^2$ .....(1), and $bx + y^2 = (s+y)^2$ .....(2).

From (1) and (2) we readily find

$$x = \frac{rs(r-s)}{as - br}, \text{ and } y = \frac{lr^2 - as^2}{2(as - br)}.$$

Substituting these values in $cx+y^2$, it becomes

$$\frac{crs(r-s)}{as-br}+\frac{(br^2-as^2)^2}{4(as-br)^2},$$

which is required to be a square; that is, we must have

$$4crs(r-s)(as-br)+(br^2-as^2)^2 = \text{a square}.$$

The above reduces to

$$b^2r^4-4bcr^3s+(4ac+4bc-2ab)r^2s^2-4acrs^3+a^2s^4 = \text{a square}.$$

Assume the above $= \left\{br^2-2crs+\left(\frac{2ac-2c^2}{b}+2c-a\right)s^2\right\}^2$.

Squaring this expression and comparing it with the above, we have

$$-4acr+a^2s = -4c\left(\frac{2ac-2c^2}{b}+2c-a\right)r+\left(\frac{2ac-2c^2}{b}+2c-a\right)^2 s.$$

By reducing this, we find $\quad r=\frac{a+b-c}{2b}s.$

We give the following numerical examples, remarking that p and q must be so assumed as to make x positive.

1. Take $p=2$ and $q=1$; then $a=7$, $b=1$, $c=5$; whence $r=\frac{3}{2}s$.

Take $s=2$, then $r=3$; whence $x=\frac{6}{11}$, and $y=-\frac{19}{22}$.

This gives $\frac{264}{361}, \frac{1320}{361}, \frac{1848}{361}$ for the roots of the required squares.

2. As another example, take $p=7$ and $q=4$; then we get $a=89$, $b=23$, $c=65$; which gives $r=\frac{47}{46}s$. Take $s=46$, then $r=47$, $x=\frac{94}{131}$, $y=-\frac{5979}{262}$; whence we obtain for the three roots, $\frac{1132888}{(5979)^2}, \frac{3201640}{(5979)^2}, \frac{4383784}{(5979)^2}$.

3. For a third example, take $p=5$, and $q=2$; this gives $a=41$, $b=-1$, $c=29$; or taking b *positive*, as we are at liberty to do, we shall have $a=41$, $b=1$, $c=29$; whence we find $r=\frac{13}{2}s$. If now we take $s=2$, we have $r=13$; whence $x=\frac{286}{69}$, $y=\frac{5}{138}$, $\frac{x}{y^2}=\frac{78936}{25}$. From these values we find the roots of the three squares to be 3157·44, 91565·76, 129455·04.

Another solution may be obtained as follows :—

Put $\quad a=2pq+(p^2-q^2)$, $b=p^2+q^2$, $c=2pq-(p^2-q^2)$;

then will a^2, b^2, c^2 be three square numbers in arithmetical progression. Now let $\frac{ax}{y^2}, \frac{bx}{y^2}, \frac{cx}{y^2}$ be the roots of the three squares required in the question; then we have to determine x and y so that $ax+y^2$, $bx+y^2$, $cx+y^2$ may all be squares.

Assume $\quad ax+y^2=l^2$, $bx+y^2=m^2$, $cx+y^2=n^2$;

then $\quad \frac{l^2-y^2}{a}=\frac{m^2-y^2}{b}=\frac{n^2-y^2}{c}=x$ $(=rst$ suppose);

therefore $\quad l^2-y^2=arst$, $m^2-y^2=brst$, $n^2-y^2=crst$.

Assume $l+y=ar$, $l-y=st$, $m+y=bs$, $m-y=rt$, $n+y=ct$, $n-y=rs$.

We must now have $2y = ar-st = bs-rt = ct-rs$.

From this $$r = \frac{s(t+b)}{t+a} = \frac{t(s+c)}{s+a};$$

whence $$(t+b)s^2 + (ab-t^2)s - ct(t+a) = 0,$$

therefore $(t^2-ab)^2 + 4ct(t+a)(t+b)$ must be a square. This reduces to

$$t^4 + 4ct^3 + 2(2ac+2bc-ab)t^2 + 4abct + a^2b^2;$$

which assume $= (t^2+2ct+2ac+2bc-2c^2-ab)^2$;

then, by reducing this, we find $t = \frac{1}{2}(c-b-a)$.

By symmetry, we have $s = \frac{1}{2}(b-c-a)$, $r = \frac{1}{2}(a-b-c)$.

Also, $$y = \tfrac{1}{2}(ar-st) = \tfrac{1}{8}(a^2+b^2+c^2-2ab-2ac-2bc);$$

whence $$\frac{x}{y^2} = \frac{8(a+c-b)(a+b-c)(a-b-c)}{(2ab+2ac+2bc-a^2-b^2-c^2)^2}.$$

Therefore the roots of the three required squares will be

$$\frac{8a(a+c-b)(a+b-c)(a-b-c)}{(2ab+2ac+2bc-a^2-b^2-c^2)^2}, \quad \frac{8b(a+c-b)(a+b-c)(a-b-c)}{(2ab+2ac+2bc-a^2-b^2-c^2)^2},$$

$$\frac{8c(a+c-b)(a+b-c)(a-b-c)}{(2ab+2ac+2bc-a^2-b^2-c^2)^2},$$

where a, b, c must, to satisfy (α), be the roots of three square numbers in arithmetical progression, such that, taking them all positively, the greatest root must *exceed* the sum of the other two.

The results given in the former solution may be readily obtained from this.

If the greatest root be *less* than the sum of the other two, the result would satisfy (β), that is, it would give three square numbers in arithmetical progression, such that, if each of their square roots be taken from unity, the three remainders shall be rational squares.

For example, take $p=3$ and $q=2$; then $a=17$, $b=13$, $c=7$; and we find for the three roots

$$\frac{ax}{y^2} = -\frac{103224}{(355)^2}, \quad \frac{bx}{y^2} = -\frac{78936}{(355)^2}, \quad \frac{cx}{y^2} = -\frac{42504}{(355)^2}.$$

Other answers may be readily found.

II. *Solution by the* Proposer.

Let x, y, z be the roots of the three required square numbers, then these have to satisfy the conditions

$$x^2+z^2 = 2y^2, \quad x+1 = m^2, \quad y+1 = n^2, \quad z+1 = p^2 \quad \ldots\ldots (1, 2, 3, 4),$$

where m, n, p are rational numbers. From (1), by the aid of (2), (3), (4), we have

$$\frac{x+y}{y+z} = \frac{y-z}{x-y}, \quad \text{or} \quad \frac{m^2+n^2-2}{n^2+p^2-2} = \frac{(n+p)(n-p)}{(m+n)(m-n)} \quad \ldots\ldots (5).$$

Assume the relations $m+n = r(n+p)$, $m-n = s(n-p)$;

then from these we obtain

$$\frac{m}{r-s+2rs} = \frac{n}{r+s} = \frac{p}{2-r+s} \left(= \frac{1}{k} \text{ suppose}\right) \quad \ldots\ldots (6).$$

Substituting these values in (5), it becomes

$$k^2 = (r-s+rs+1)^2+(rs+1)^2+4\frac{r-s}{rs-1} \quad \ldots\ldots\ldots\ldots (7);$$

and we have to determine r and s so that the second side of (7) may be a perfect square. In order to effect this, let $rs-1=f$; then, putting for shortness' sake $g=\frac{1}{2}\left(f+2+\frac{2}{f}\right)$, (7) becomes

$$k^2 = (r+s)^2+4g(r-s+f) \quad [=(r+s-2g)^2 \text{ suppose}] \quad \ldots\ldots\ldots (8).$$

From (8) we find $r=\frac{1}{2}(g-f)$; and thence, by the aid of (6), (2), (3), (4), we obtain the following general expressions for x, y, z, the roots of the required squares:

$$\frac{4(rs+r-g)(rs-s+g)}{(r+s-2g)^2}, \quad \frac{4g(r+s-g)}{(r+s-2g)^2}, \quad \frac{4(1-r+g)(1+s-g)}{(r+s-2g)^2},$$

where $g=\frac{1}{2}f+1+\frac{1}{f}$, $r=\frac{1}{2}(g-f)$, and $s=\frac{1+f}{r}$,

f being any rational number whatever.

Taking $f=1$, we obtain the numbers given as the results to satisfy the *first* condition (α) when the question was originally proposed, viz.

$$\frac{264}{361}, \quad \frac{1320}{361}, \quad \frac{1848}{361}.$$

Taking $f=\frac{1}{4}$, we find

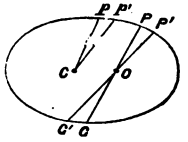
$$-x=\frac{42504}{(355)^2}, \quad -y=\frac{78936}{(355)^2}, \quad -z=\frac{103224}{(355)^2};$$

hence these fractions are the roots of three square numbers in arithmetical progression which satisfy the *second* condition (β) in the question, that is, they are such that, if the roots be taken from unity, the three remainders will be rational squares.

2486. (Proposed by G. O. HANLON.)—Find a point on an ellipse such that the normal produced to meet the curve again may cut off a maximum area.

I. *Solution by* PROFESSOR WHITWORTH.

Let PG be the required normal meeting the conic again in G. Let P′ be a point on the curve adjacent to P, and P′G′ the normal chord at P′. Let O be the point of intersection of these two normals. Then ultimately, when P′ coincides with P, O is the centre of curvature at P.



Since PG is the normal which divides the area the most unequally, therefore the greater area cut off by PQ is a maximum. Hence ultimately area PP′O = area GG′O. But these areas are ultimately in the ratio of the rectangles OP.OP′ and OG.OG′.

Therefore $OP \cdot OP' = OG \cdot OG'$ ultimately. But if $Cp \cdot Cp'$ be semi-diameters parallel to OP and OP', we have

$$OP \cdot OG : OI' \cdot OG' = Cp^2 : Cp'^2;$$

therefore $OP^2 : OG'^2 = Cp^2 : Cp'^2$, or $OP : OG' = Cp : Cp'$.

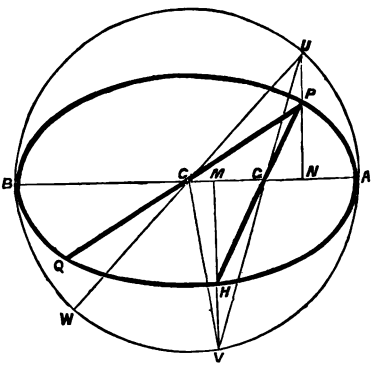
But ultimately $Cp = Cp'$; therefore ultimately $OP = OG'$ or $OP = OG$. Hence PG is the diameter of curvature and the common chord of the ellipse and circle of curvature. Hence PG and the tangent at P make equal angles with the axis, viz., angles of 45°. Therefore the normal required is a normal inclined at an angle of 45° to the axis.

COROLLARY.—The normal which cuts off the smallest closed area from any conic is inclined at an angle of 45° to the axes.

[A solution by Mr. JENKINS has been given on p. 56 of Vol. X. of the *Reprint*.]

II. *Solution by the* EDITOR.

Let C be the centre of the ellipse; PCQ the diameter, and PGH the normal chord through P; NPU and MHV the ordinates at P and H, meeting the circle circumscribing the ellipse in U and V; and UCW a diameter of the circle through U.



Then it may be readily shown that U, G, V are in the same straight line; and by a well known property we have

$$\frac{\text{area AGP}}{\text{area AGU}} = \frac{AGH}{AGV} = \frac{APH}{AUV} = \frac{PHQ}{UVW} = \frac{b}{a},$$

a and b being the semi-axes of the ellipse. Hence, putting S for the area of PHQ, and ϕ for the angle VCW, we have

$$S = \text{area PHQ} = \frac{b}{a}(UVW) = \tfrac{1}{2}ab(\phi + \sin\phi).$$

Now $$\frac{UN}{PN} = \frac{a}{b}, \quad \text{and} \quad \frac{CN}{GN} = \frac{a^2}{b^2};$$

hence $$\tan UGN = \frac{a}{b}\tan\theta, \quad \text{and} \quad \tan UCN = \frac{b}{a}\tan\theta;$$

therefore $$\tan\tfrac{1}{2}\phi = \tan CUG = \tan(UGN - UCN) = \frac{a^2-b^2}{2ab}\sin 2\theta.$$

Hence $S = \tfrac{1}{2}ab(\phi + \sin\phi)$, where $2ab\tan\tfrac{1}{2}\phi = (a^2-b^2)\sin 2\theta$.

When S is a maximum, we must have

$$\frac{dS}{d\theta} = \frac{dS}{d\phi}\cdot\frac{d\phi}{d\theta} = ab(a^2-b^2)(1+\cos\phi)\cos 2\theta = 0;$$

whence we find the angle $PGN = \theta = 45°$.

Hence we see that the normal chord PGH, which makes an angle of 45° with the major axis, divides the ellipse into two segments, one of which (APH) is a *minimum*, and the other (BPH) a *maximum*.

2527. (Proposed by J. WILSON.)—Find the locus of the intersection of two tangents to a circle, the chord of contact of which subtends a right angle at a fixed point; and also of the middle points of a system of chords of a circle, which subtend a right angle at a fixed point; and thence show that the envelope of a system of chords of a conic, which subtend a right angle at the focus, is another conic of coincident focus; and that the locus of the middle points of a system of chords of a conic which subtend a right angle at a fixed point is another conic.

Solution by the Rev. J. WOLSTENHOLME, M.A.

Taking the fixed point for origin, let equation of circle be $(x-c)^2+y^2=a^2$, and let $lx+my=1$ be a chord. The lines joining the points of intersection to the origin have for their equation

$$x^2+y^2-2cx(lx+my)+(c^2-a^2)(lx+my)^2=0;$$

and if they be at right angles,

$$2-2cl+(c^2-a^2)(l^2+m^2)=0.$$

We may then write the equation of two parallel straight lines satisfying the condition in the form

$$2(lx+my)^2-2cl(lx+my)+(c^2-a^2)(l^2+m^2)=0,$$

and the envelope is therefore

$$(2x^2-2cx+c^2-a^2)(2y^2+c^2-a^2)=(2xy-cy)^2, \text{ or } \frac{(2x-c)^2}{2a^2-c^2}+\frac{2y^2}{a^2-c^2}=1;$$

a conic whose foci are the origin and the centre of the circle. The middle point of the chord being the foot of the perpendicular from the centre, lies on the auxiliary circle of this ellipse, *i.e.* on the circle $(2x-c)^2+4y^2=2a^2-c^2$. The locus of the intersection of tangents at the extremities of such chords is the reciprocal polar of this conic with respect to a circle whose centre is a focus, and is therefore a circle. Of course the envelope of a system of chords of a conic subtending a right angle at any point is a conic with that point as focus; as we get at once by reciprocating the property of the director circle of a conic.

If the equation of any conic be $ax^2+by^2+2hx+2ky+1=0$, and the straight line $lx+my=1$ subtend a right angle at the origin,

$$a+b+2(hl+km)+l^2+m^2=0,$$

and the coordinates of the middle point are

$$\frac{X}{kml-hm^2+bl}=\frac{Y}{hlm-kl^2+am}=\frac{1}{bl^2+am^2}=\frac{lX+mY}{bl^2+am^2}$$

$$=\frac{amX-blY}{(am^2+bl^2)(kl-hm)},$$

or

$$lX+mY=1, \quad l(k+lY)=m(h+aX);$$

therefore

$$\frac{l}{h+aX}=\frac{m}{k+bY}=\frac{1}{aX^2+lY^2+hX+kY};$$

and the locus of the middle points is apparently a curve of the fourth degree.

2769. (Proposed by A. MARTIN.)—From a point, taken at random within a triangle, perpendiculars are drawn on the sides; find the probability that the triangle formed by joining the feet of these perpendiculars is acute-angled.

Solution by the Rev. J. WOLSTENHOLME, M.A.

In an acute-angled triangle ABC, if O be the centre of the circumscribed circles, and arcs of circles be described on each side touching the radii from O to the ends of the side, the points must lie without each of the arcs. The result does not seem to be very neat, and it becomes more complicated for an obtuse-angled triangle.

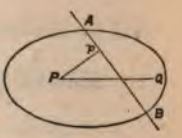
2763. (Proposed by W. S. BURNSIDE, M.A.)—Give a geometrical interpretation for the relation between the invariants of two conics, viz., $\Theta\Theta' = \Delta\Delta'$.

Solution by the Rev. J. WOLSTENHOLME, M.A.

The interpretation is given in Salmon's *Conic Sections*. It is the condition that the 8 points of contact of the common tangents to the two conics may lie on two straight lines; in which case also the 8 tangents to the two conics at their common points will pass through two points, four (two from each conic) through one point and four through the other. An example is any ellipse and a concentric rectangular hyperbola passing through the foci of the ellipse; but the common tangents are unreal. Another example of two circles of radii 1, 7, with distances between centres 10, has the common tangents real, but the common points impossible.

2109. (Proposed by Capt. CLARKE, R.E., F.R.S.)—Two lines are drawn at random across a convex closed curve; determine the chance of their intersecting.

Solution by the PROPOSER.



Let P be a point within the curve, PQ a fixed line. Let AB be any line crossing the curve, the normal to which, Pp, makes an angle θ with PQ, and let the intercepted chord AB $= u$. Let a second line, whose normal makes an angle θ', be drawn intersecting AB; the number of such lines is $\pm u \sin(\theta - \theta')$ according as θ is greater or less than θ': the values of θ, θ' to range from 0 to π. The number of lines then that intersect AB is

$$u\int_0^{\theta} \sin(\theta-\theta')\,d\theta' + u\int_{\theta}^{\pi} \sin(\theta'-\theta)\,d\theta' = u(1-\cos\theta) + u(1+\cos\theta),$$

or $2u$. If $Pp=p$, the number of lines intersecting all lines parallel to AB is

$$2\int udp = 2A,$$

where A is the area of the curve; and consequently the total number of pairs of lines that intersect is $2\pi A$. It remains to find the total number of lines that cross the curve. Let p, p' be the perpendiculars from P upon a pair of parallel tangents to the curve, then $p+p'$ is the number of lines which cross parallel to those tangents, and the total number is therefore

$$\int_0^\pi (p+p')\,d\theta = P,$$

where P is the perimeter of the curve. The required probability of intersection is, therefore, $\frac{2\pi A}{P^2}$.

2773. (Proposed by MATTHEW COLLINS, B.A.)—When a given angle rolls upon a fixed parabola, the locus of its vertex is well known to be a hyperbola having the same focus and directrix as the parabola. Conversely, when a given parabola rolls within a fixed angle, show that its focus and vertex describe lines of the fourth and sixth orders respectively; find the *actual equations* of both these curves, and thence show that the vertex of the fixed angle is a conjugate point on the former curve, and that the equation of the latter, when the given angle is a right angle, becomes

$$x^2y^2(x^2+y^2+3a^2) = a^6 \quad \text{or} \quad x^{\frac{2}{3}}y^{\frac{2}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}}) = a^2,$$

the sides of the given angle being the axes, and a equal to the distance of the focus from the vertex.

Solution by the Rev. J. WOLSTENHOLME, M.A.

The equation of a parabola touching the axes of x, y, which are inclined at angle ω, being $\left(\frac{x}{h}\right)^{\frac{1}{2}}+\left(\frac{y}{k}\right)^{\frac{1}{2}}=1$, the coordinates of the focus are

$$\frac{X}{k} = \frac{Y}{h} = \frac{hk}{h^2+k^2+2hk\cos\omega},$$

and the equation of the directrix is

$$x(h+k\cos\omega)+y(k+h\cos\omega) = hk\cos\omega.$$

Hence, if $4a$ be the latus rectum, we have

$$2a = \frac{\dfrac{hk^2(h+k\cos\omega)+h^2k(k+h\cos\omega)}{h^2+k^2+2hk\cos\omega} - hk\cos\omega}{\left\{(h+k\cos\omega)^2+(k+h\cos\omega)^2-2(h+k\cos\omega)(k+h\cos\omega)\cos\omega\right\}^{\frac{1}{2}}}$$

$$= \frac{2h^2k^2\sin^2\omega}{(h^2+k^2+2hk\cos\omega)^{\frac{3}{2}}};$$

whence the equation of the locus of the focus is

$$a^2(X^2+Y^2+2XY\cos\omega) = X^2Y^2\sin^2\omega,$$

a curve of the fourth degree having a conjugate circular point at the origin, and having the four asymptotes $X = \pm a\operatorname{cosec}\omega$, $Y = \pm a\operatorname{cosec}\omega$. The vertex is most readily found from being the point of contact of a tangent parallel to the directrix, which gives at once, if (X, Y) be the vertex,

$$\frac{(kY)^{\frac{1}{2}}}{h+k\cos\omega} = \frac{(hX)^{\frac{1}{2}}}{k+h\cos\omega} = \frac{hk}{h^2+k^2+2hk\cos\omega} = \frac{a}{\sin^2\omega}\,\frac{(h^2+k^2+2hk\cos\omega)^{\frac{1}{2}}}{hk};$$

three equations from which to eliminate h, k. In the case where $\omega = \frac{1}{3}\pi$, we have

$$\frac{X}{k^3} = \frac{Y}{h^3} = \frac{hk}{(h^2+k^2)^2} = \frac{a^2(h^2+k^2)}{h^3k^3} \quad \text{or} \quad hX^{\frac{1}{3}} = kY^{\frac{1}{3}} = (X^{\frac{2}{3}}+Y^{\frac{2}{3}})^2,$$

and
$$a^2 = \frac{k^3Y}{h^2+k^2} = \frac{(X^{\frac{2}{3}}+Y^{\frac{2}{3}})^6}{(X^{\frac{2}{3}}+Y^{\frac{2}{3}})^4(X^{-\frac{2}{3}}+Y^{-\frac{2}{3}})} = X^{\frac{2}{3}}Y^{\frac{2}{3}}(X^{\frac{2}{3}}+Y^{\frac{2}{3}}),$$

or
$$a^6 = X^2Y^2(X^2+Y^2+3a^2).$$

2774. (Proposed by J. J. WALKER, M.A.) — A central conic $(b^2x^2 \pm a^2y^2 = a^2b^2)$ is turned in its own plane about its centre through a right angle; prove that the locus of the intersection of the normal at any point on the given conic with the same line in its new position is one of the two sextic curves

$$(x^2+y^2)^2\left\{a^2(x\pm y)^2 \pm b^2(x\mp y)^2\right\} = (a^2\mp b^2)^2(x^2-y^2)^2.$$

Quaternion Solution by W. H. LAVERTY, B.A.

The vector equations to the two normals are

$$\rho = \quad \alpha\cos\tau + \quad \beta\sin\tau + x(b^2\alpha\ \cos\tau + a^2\beta\ \sin\tau),$$

and
$$\rho' = \frac{a}{b}\beta\cos\tau - \frac{b}{a}\alpha\sin\tau + y(ba\beta\cos\tau - ab\alpha\sin\tau);$$

and that these may coincide, we have

$$-x\,.\,ab(a^2\sin^2\tau + b^2\cos^2\tau) = ab + (b^2-a^2)\sin\tau\cos\tau.$$

Whence, if $a^2\sin^2\tau + b^2\cos^2\tau = P$, the equation to the locus becomes

$$ab\,.\,P\,.\,\rho = \sin\tau\,.\cos\tau\,(a^2-b^2)\left\{b\,.\,\alpha(a\sin\tau + b\cos\tau) + a\,.\,\beta(a\sin\tau - b\cos\tau)\right\}.$$

To transform this to a Cartesian equation, we have

$$P\,.\,x = \sin\tau\,.\cos\tau\,(a^2-b^2)(a\sin\tau + b\cos\tau),$$
$$P\,.\,y = \sin\tau\,.\cos\tau\,(a^2-b^2)(a\sin\tau - b\cos\tau);$$

therefore
$$P^2\left\{a^2(x-y)^2 + b^2(x+y)^2\right\} = 4a^2b^2\sin^2\tau\cos^2\tau\,(a^2-b^2)^2,$$
$$(x^2+y^2)^2P^2 = 4\sin^4\tau\cos^4\tau\,(a^2-b^2)^4,$$
$$(x^2-y^2)^2P^4 = 16a^2b^2\sin^6\tau\cos^6\tau\,(a^2-b^2)^4,$$

whence the required equation. Similarly for the hyperbola.

2244. (Proposed by the late W. LEA.)—Form 11 symbols into sets, 5 symbols in a set, so that every combination of 4 symbols shall appear once in the sets.

2712. (Proposed by the late W. LEA.)—Form 15 symbols into sets, 5 symbols in each set, so that every combination of 4 symbols shall appear once in the sets.

2755. (Proposed by the late W. LEA.)—Form 16 symbols into sets, 4 symbols in each set, so that every triad in the symbols shall appear once in the sets.

Solution by the Rev. T. P. KIRKMAN, M.A., F.R.S.

The problem of 4-plets exhausting the triads of 16 is a case of the more general theorem which I gave in the *Cambridge and Dublin Mathematical Journal*, 1853, p. 42, thus: "2^n young ladies can all walk together in fours day by day till every three have walked together." The proof is very simple. If $n=2m$, we take for our 4^m symbols the repeating variations that can be made of them in 4-plets. Then every triplet is completed into a 4-plet by the rule that no 4-plet shall have only three rth places either like or unlike. Thus the triplet *abbadd . bbcaab . cdaaaa* ($m=6$) is completed into a 4-plet of sixes by the six *dddadc*. When $n=2m+1$, we join to the 4^m symbols made with *abcd* the 4^m made with abcd, and add this rule, that no 4-plet shall have only three Italic or only three Roman symbols; thus, if out of 512 young ladies the three (abbd), (*abbb*), and (*adaa*) choose to walk together on any day, their companion must be (adac), and I have shown that any triplet whatever will determine the entire arrangement for the day.

To form 5-plets with eleven elements so as once to exhaust the 4-plets (Question 2244), we first form the 12 5-plets that contain the two elements $0a$; thus

0*a*123	0*a*147	0*a*159	0*a*168	
0*a*456	0*a*258	0*a*267	0*a*249	(0*a*)
0*a*789	0*a*369	0*a*348	0*a*357	

which are obtained by properly reading any one of the four squares of triplets in figures. We call this set of 12 (0*a*). There must be in the solution a set of 12 (01), another (02), &c., another (12), &c., such that, if we form them all, we shall construct all the 11.6 quintuplets 10 times. Denote by (*a*1, 4678) (0*a*) the operation on (0*a*) with the two circles *a*1, 4678, that is, the exchange of *a* and 1, and the putting 6 for 4, 7 for 6, 8 for 7, and 4 for 8. Then we find that

(01) = (*a*1, 4678) (0*a*),		(*a*1) = (01, 8764) (0*a*),	
(02) = (*a*2, 5489) (0*a*),		(*a*2) = (02, 9845) (0*a*),	
(03) = (*a*3, 6597) (0*a*),		(*a*3) = (03, 7956) (0*a*),	
(07) = (*a*7, 1345) (0*a*),		(*a*4) = (04, 2197) (0*a*),	
(08) = (*a*8, 2156) (0*a*),		(*a*5) = (05, 3278) (0*a*),	
(09) = (*a*9, 3264) (0*a*),		(*a*6) = (06, 1389) (0*a*),	
(04) = (*a*4, 7912) (0*a*),		(*a*7) = (07, 5431) (0*a*),	
(05) = (*a*5, 8723) (0*a*),		(*a*8) = (08, 6512) (0*a*),	
(06) = (*a*6, 9831) (0*a*),		(*a*9) = (09, 4623) (0*a*).	

We have now every quintuplet containing 0 and all containing a, in number $12+18+18=48$.

If $\theta = (4678)$, $\theta^{-1} = \theta^3 = (8764)$, and $\theta^2 = (47, 68)$; and the operation

(0a, 47, 68) on (01) will give (a1), and *vice versâ*. (10) and (1a) can be formed by reading horizontally, vertically, and parallel to the diagonals, the two triplet squares in the right members following:

$$(10) = 10\,.\,a23 \qquad (1a) = 1a\,.\,023$$
$$\begin{array}{rr} 657 & 854 \\ 849 & 679 \end{array}$$

and we obtain 1a, 12, 13, &c from (10), thus

$$(1a) = (0a, 47, 68)\,(10), \qquad (16) = (06, 29, 8a)\,(10),$$
$$(12) = (02, 96, 54)\,(10), \qquad (15) = (05, 38, 42)\,(10),$$
$$(13) = (03, 85, 79)\,(10), \qquad (17) = (07, a4, 93)\,(1a),$$
$$(18) = (08, 53, a6)\,(10),$$
$$(14) = (04, 7a, 25)\,(10),$$
$$(19) = (09, 62, 37)\,(10).$$

We have now every quintuplet containing 1. Next we write, by (2a) = (0a, 94, 85) (20), $(20) = 20\,.\,1a3 \qquad (2a) = 2a\,.\,103$

$$\begin{array}{rr} 846 & 596 \\ 795 & 748 \end{array}$$

and we obtain, omitting (2a) and (21) already found,

$$(23) = (03, 56, 47)\,(20), \qquad (27) = (07, 81, 34)\,(20),$$
$$(28) = (08, 17, 5a)\,(20), \qquad (29) = (09, 4a, 16)\,(20),$$
$$(24) = (04, a9, 73)\,(20), \qquad (25) = (05, 63, a8)\,(20).$$
$$(26) = (06, 35, 91)\,(20),$$

This gives us every 5-plet containing 2. Next, we write (30) and (3a) = (0a, 69, 57)(30),

$$(30) = 30\,.\,12a \qquad (3a) = 3a\,.\,120$$
$$\begin{array}{rr} 495 & 467 \\ 687 & 985 \end{array}$$

The process is continued thus:—

$$(34) = (04, 27, 61)\,(30), \qquad (36) = (06, 9a, 14)\,(30),$$
$$(39) = (09, a6, 83)\,(30), \qquad (38) = (08, 51, 29)\,(30),$$
$$(35) = (05, 18, 7a)\,(30), \qquad (37) = (07, 42, a5)\,(30),$$

which gives all containing 3.

We now add the 54 following to the 12 above formed with 0a:—

01657	02795	a1854	a2748	12947	17958
01849	02347	a1679	a2397	12856	23875
01254	02365	a1257	a2368	12376	23496
01379	02389	a1349	a2345	12348	24765
01358	03495	a1356	a3467	12359	24589
01629	03687	a1829	a3958	13547	26789
01278	04587	a1246	a4759	13689	34789
01346	04679	a1378	a4986	14569	34568
02846	05689	a2596	a5876	14867	35679

If it is possible to form quintuplets with 15 letters so as to exhaust the quadruplets, there must be 26 containing the final letters de. These are easily formed by cyclically permuting the 13 elements 123 .... 90abc under 125 and 138. This gives us

(de) = de125, de236, ... de138, de249 .....

There must be a set (d1) of 26, and another (e1). We readily find

$$(1e) = (1d, 2358, 7a90)\,(de),$$
$$(1d) = (1e, 8532, 09a7)\,(de),$$

results so far analogous to the preceding.

But I cannot find the substitutions that will correctly give (2*e*), (2*d*), &c. I shall be agreeably surprised to see this problem solved. It would be pleasant to find it to be a simple evolution of the famous fifteen young ladies whom I had the honour of introducing to the planet for the first time in the *Lady's and Gentleman's Diary* for 1850. To my original puzzle of seven arrangements of five triads so that every pair should once walk abreast, Professor SYLVESTER added a second, to make the school of 15 walk out every day in the quarter in 13.7 columns of triplets, till every three have once walked abreast. I gave, at the time, one solution of Prof. SYLVESTER'S question in the *Cambridge and Dublin Mathematical Journal*. It would be a memorable sight to see his 13.7 columns of 5 triplets made to perform the right-face, so as to present the 13.7.3 quintuplets here required. Any field-marshal could transform by a word the column of 5.3 into a dangerous phalanx of 3.5; but I fear that a couple of generalissimos would find it a heavy strain on their tactics to turn out day by day for the quarter the exact quintuplets of Mr. LEA.

2893. (Proposed by M. W. CROFTON, F.R.S.)—If there be (n) quantities $a, b, c, d,$ &c., each of which takes independently a given number of values $a_1 a_2 a_3 \ldots . b_1 b_2 b_3 \ldots .$ &c. (the number may be different for each), if we put

$$\Sigma(a) = a + b + c + d + \&c.,$$

and if for shortness we denote "the mean value of x" by $\mathrm{M}(x)$, prove that

$$\mathrm{M}[\Sigma(a)] = \mathrm{M}(a) + \mathrm{M}(b) + \mathrm{M}(c) + \&c. = \Sigma[\mathrm{M}(a)],$$
$$\mathrm{M}[\Sigma(a)]^2 = \{\Sigma[\mathrm{M}(a)]\}^2 - \Sigma[\mathrm{M}(a)]^2 + \Sigma[\mathrm{M}(a^2)].$$

Solution by W. H. H. HUDSON, M.A.

Here a may have any of r values, $a_1, a_2, a_3 \ldots$; b any of s values, $b_1, b_2, b_3 \ldots$; c any of t values, $c_1, c_2, c_3 \ldots$; &c. &c.;

therefore $a+b+c+\ldots$ may have any of the $rst\ldots$ values

$$a_1+b_1+c_1+\ldots,\quad a_2+b_2+c_2+\ldots,\quad \ldots\ldots,$$

whereof a_1 occurs in $st\ldots$ sets, b_1 in $rt\ldots$ sets, and so on;

therefore $\dfrac{\Sigma(a+b+c)\ldots}{rst\ldots} (= \mathrm{M}\Sigma a) = \dfrac{st\ldots(a_1+a_2+\ldots)+t\ldots(b_1+b_2+\ldots)+\ldots}{rst\ldots}$

$$= \frac{a_1+a_2+\ldots}{r} + \frac{b_1+b_2+\ldots}{s} + \ldots = \mathrm{M}a + \mathrm{M}b + \mathrm{M}c + \ldots = \Sigma \mathrm{M}a.$$

Again, $(\Sigma a)^2$ may have $rst\ldots$ values, of which the type is

$$\ldots a^2 + b^2 + c^2 + \ldots + 2ab + 2bc + \ldots$$

a_1^2 will occur in $st\ldots$, b_1^2 in $rt\ldots$, and so on; $a_1 b_1$ will occur in $t\ldots$, $b_1 c_1$ in $r\ldots$, and so on

Therefore $\mathrm{M}(\Sigma a)^2 =$

$$\frac{st\ldots(a_1^2+a_2^2+\ldots)+rt\ldots(b_1^2+\ldots)+2t\ldots(a_1b_1+\ldots)+2r\ldots(b_1c_1+\ldots)}{rst}$$

$$= \frac{a_1^2+a_2^2+\ldots}{r} + \frac{b_1^2+b_2^2+\ldots}{s} + \ldots + \frac{2(a_1b_1+a_2b_2+\ldots)}{rs} + \frac{2(b_1c_1+\ldots)}{st} + \ldots$$

$$\text{Now } \left(\frac{a_1+a_2+\dots}{r}+\frac{b_1+b_2+\dots}{s}+\frac{c_1+c_2+\dots}{t}+\dots\right)^2$$
$$-\left\{\left(\frac{a_1+a_2+\dots}{r}\right)^2+\left(\frac{b_1+b_2+\dots}{s}\right)^2+\dots\right\}$$
$$=\frac{2(a_1b_1+\dots)}{rs}+\frac{2(b_1c_1+\dots)}{st}+\dots;$$

$$\text{therefore } \mathrm{M}(\Sigma a)^2=\mathrm{M}(a^2)+\mathrm{M}(b^2)+\dots$$
$$+\left\{\left(\frac{a_1+a_2+\dots}{r}\right)+\left(\frac{b_1+b_2+\dots}{s}\right)+\dots\right\}^2$$
$$-\left(\frac{a_1+a_2+\dots}{r}\right)^2-\left(\frac{b_1+b_2+\dots}{s}\right)^2-\dots$$
$$=\Sigma\mathrm{M}a^2+(\Sigma\mathrm{M}a)^2-\Sigma(\mathrm{M}a)^2.$$

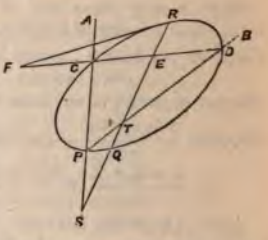
2887. (Proposed by ARCHER STANLEY.)—1. If a point P on a conic be connected with any two fixed points A and B in its plane, all chords which are divided harmonically by PA and PB will be concurrent.

2. The locus of the point of concurrence when P is variable is another conic which has double contact with the given one on AB.

Solution by F. D. THOMSON, M.A.

1. Let QR be the chord divided harmonically by PA, PB. Then, constructing as in the figure,

$$-1=\mathrm{P}\{\mathrm{SQTR}\}=\mathrm{P}\{\mathrm{CQDR}\}$$
$$=\mathrm{R}\{\mathrm{CEDF}\};$$



therefore F is the pole of QR, and therefore QR passes through the pole of CD, *i.e.* through a fixed point.

2. If O be the pole of CD, then when P varies the locus of O is a conic. For consider the points in which the locus of O is cut by the tangent to the given conic through A. Let the tangent through A have L for its point of contact. Let BL meet the conic in M, AM meet the conic in N. Then the poles of LM and LN are the points on the required locus upon the tangent AL. Hence the locus is a conic. Also the tangents to the given conic at the points where AB meets it are seen to be tangents to the required locus. Hence the locus is a conic having double contact with the given conic at two points on AB.

The particular case when A and B are the circular points is given by Salmon, p. 257.

2728. (Proposed by W. S. McCay, B.A.)—Given three planes and their poles with regard to a system of quadrics, the locus of centre is a right line.

Solution by the Proposer.

For the lines joining the vertices of a tetrahedron to the corresponding vertices of its polar tetrahedron belong to the same system of generators of an hyperboloid of one sheet. (Salmon's *Geometry of Three Dimensions*, p. 179.) Hence, the centre being the pole of the plane at infinity, we see that if through each of the given poles we draw lines parallel to the intersection of the other two planes, the hyperboloid of which these are generators passes through the intersection of the three planes, and the locus of centre is the generator of the same system at that point.

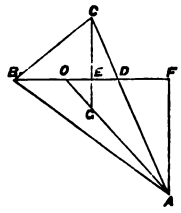
2597. (Proposed by W. H. H. Hudson, M.A.)—A right cone, whose weight may be neglected, is suspended from a point in its rim; it contains as much fluid as it can: show that the whole pressure upon its surface is

$$\tfrac{1}{3}\pi\rho g h^3 \frac{\sin\alpha\cos\theta}{\cos^2\alpha}\left\{\frac{\cos(\theta+\alpha)}{\cos(\theta-\alpha)}\right\}^{\frac{3}{2}},$$

where h and 2α are the height and vertical angle of the cone, and θ is determined from $3\sin 2\theta = 4\sin 2(\theta-\alpha)$.

Solution by Stephen Watson.

Let C be the point of suspension; ABC a vertical section of the cone; BF a horizontal line cutting AC in D, and meeting a perpendicular from A in F; and O the middle of BD. Join AO, and take $OG = \frac{1}{4}OA$; then G is the centre of gravity of the fluid; hence CG must cut OF perpendicularly in E, so that $OE = \frac{1}{4}OF$. Put $\angle CBD = \theta$, the area of the horizontal surface of the fluid $= S$, and the area of the surface of the cone in contact with the fluid $= (A)$.



Now $BE = BC\cos\theta = 2h\tan\alpha\cos\theta$,

$BF = BA\cos(\frac{1}{2}\pi-\alpha-\theta) = h\sec\alpha\sin(\theta+\alpha)$,

and $$BO = \tfrac{1}{2}BD = \frac{BC\sin BCD}{2\sin BDC} = \frac{h\sin\alpha}{\cos(\theta-\alpha)};$$

hence, by substitution in $BF - BO = 4(BE - BO)$ or $BF + 3BO = 4BE$, we have $\sin(\theta+\alpha)\cos(\theta-\alpha) + 3\sin\alpha\cos\alpha = 8\sin\alpha\cos\theta\cos(\theta-\alpha)$,

$$\therefore \quad \tfrac{1}{2}(\sin 2\theta + \sin 2\alpha) + \tfrac{3}{2}\sin 2\alpha = 4\sin\alpha\left\{\cos(2\theta-\alpha)+\cos\alpha\right\}$$
$$= 2\left\{\sin 2\theta - \sin 2(\theta-\alpha)\right\} + 2\sin 2\alpha,$$

therefore $3\sin 2\theta = 4\sin 2(\theta-\alpha)$, the condition for finding θ.

Again, if a, b ($a = OB$) be the semi-axes of the surface S, we have, by

putting $\frac{1}{2}\pi-\alpha+\theta$ for α, and α for β, in equation (12), p. 295, Vol. II., of Davies' edition of Hutton's *Course*,

$$S = \pi ab = \frac{\pi BO^2}{\cos\alpha}\left\{\cos(\theta+\alpha)\cos(\theta-\alpha)\right\}^{\frac{1}{2}} = \frac{\pi h^2\sin^2\alpha}{\cos\alpha}\cdot\frac{\cos^{\frac{1}{2}}(\theta+\alpha)}{\cos^{\frac{3}{2}}(\theta-\alpha)}.$$

Also the distance of the centre of gravity of (A) from the horizontal line BF is $\frac{1}{3}AF = \frac{1}{3}BA\sin(\frac{1}{2}\pi-\alpha-\theta) = \frac{1}{3}h\sec\alpha\cos(\theta+\alpha)$ .....(1).

Moreover, the orthogonal projections of the surfaces S and (A) upon the base of the cone must be equal, that is,

$$(A)\sin\alpha = S\cos\theta, \quad \text{therefore} \quad (A) = \frac{\cos\theta}{\sin\alpha}S;$$

hence the whole pressure upon the cone's surface is

$$\tfrac{1}{3}\rho g\times(A)\times(1) = \tfrac{1}{3}\pi\rho g h^3\frac{\sin\alpha\cos\theta}{\cos^2\alpha}\left\{\frac{\cos(\theta+\alpha)}{\cos(\theta-\alpha)}\right\}^{\frac{3}{2}}.$$

2108. (Proposed by W. K. CLIFFORD, B.A.) — Required analogues in Solid Geometry to the following propositions in Plane Geometry:—

(*a*.) The perpendiculars of a triangle meet in a point.

(*b*.) The middle points of the diagonals of a quadrilateral are in one straight line.

(*c*.) The circles whose diameters are the diagonals of a quadrilateral have a common radical axis.

(*d*.) Every rectangular hyperbola circumscribing a triangle passes through the intersection of perpendiculars.

(*e*.) Every rectangular hyperbola to which a triangle is self-conjugate passes through the centres of the four touching circles.

(*f*.) $\sin(A+B) = \sin A\cos B + \cos A\sin B$.

(*g*.) The sum of the angles of a triangle = two right angles.

(*h*.) In any triangle $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{C}$.

Solution by the PROPOSER.

I have an analogue for each of the four (*d*), (*e*), (*f*), (*g*), and more than one for each of the others obtained by extensions of Mr. GREER'S methods. *E. g.*:—

(*c*.) A straight line cuts the faces of a tetrahedron ABCD in *a*, *b*, *c*, *d*; the spheres whose diameters are A*a*, B*b*, C*c*, D*d*, have a common radical axis. Hence the middle points of these four lines are in one plane.

Let a conicoid whose asymptotic cone has three generating lines at right angles be called a rectangular conicoid.

(*d*), (*e*). Every rectangular conicoid circumscribing a tetrahedron whose perpendiculars meet in a point, passes through the point. And every rectangular conicoid to which a tetrahedron is self-conjugate, passes through the centres of the eight touching spheres.

(*f*.) $$\sin(ABC) = \sin(ABD+BCD+CAD)$$
$$= \sin(BCD).\cos\widehat{AD} + \sin(CAD).\cos\widehat{BD} + \sin(ABD).\cos\widehat{CD},$$
where A, B, C, D are four lines in space, and
$$\sin^2 ABC = \begin{vmatrix} 1, & \cos\widehat{AB}, & \cos\widehat{AC} \\ \cos\widehat{AB}, & 1, & \cos\widehat{BC} \\ \cos\widehat{AC}, & \cos\widehat{BC}, & 1, \end{vmatrix}$$

(*g*.) In the triangle case this should be written
$$(BC)+(CA)+(AB) = 0.$$
The analogue is then obviously
$$(BCD)-(CDA)+(DAB)-(ABC) = 0,$$
A, B, C, D being any four planes.

(*h*.) In any tetrahedron,
$$\frac{AC.DB}{\sin\widehat{AC}.\sin\widehat{DB}} = \frac{abcd}{V^2} = \frac{a}{\cos A} = \frac{V^6}{(abcd)^2}.\cos A\cos B\cos C\cos D,$$
where a, b, c, d are the faces, and $\cos^2 A = \begin{vmatrix} 1, & \cos\widehat{BC}, & \cos BD \\ \cos\widehat{BC}, & 1, & \cos\widehat{CD} \\ \cos\widehat{BD}, & \cos\widehat{CD}, & 1 \end{vmatrix}$
($\widehat{AB}$, &c. denoting angles between planes).

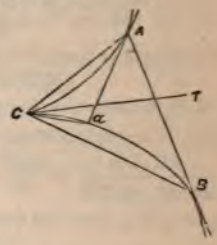
2651. (Proposed by W. S. BURNSIDE, M.A.)—Determine the form of the solution of the differential equation
$$2f(x)\frac{d^2U}{du^2} + 3f'(x)\frac{dU}{du} + \left\{f''(x) \pm n^2\right\}U = 0 \quad \ldots\ldots\ldots (1).$$

Solution by the PROPOSER.

The differential equation
$$f(x)\frac{d^2y}{du^2} + \tfrac{1}{2}f'(x)\frac{dy}{du} \pm n^2 y = 0 \quad \ldots\ldots\ldots\ldots (2),$$
is reduced to the form $\frac{d^2y}{dt^2} \pm n^2 y = 0$ by assuming $t = \int \frac{du}{\sqrt{f(x)}}$ for the independent variable; whence, taking the lower sign,
$$y = ce^{n\int \frac{du}{\sqrt{f(x)}}} + c_1 e^{-n\int \frac{du}{\sqrt{f(x)}}}.$$
Again, differentiating the equation (2) with regard to x, and writing U for $\frac{dy}{dx}$, and $2f(x)$ for $f(x)$, we find the given equation (1), the solution of which is therefore
$$U = (fx)^{-\frac{1}{2}}\left\{ae^{n\int \frac{dx}{\sqrt{2f(x)}}} + be^{-n\int \frac{dx}{\sqrt{2f(x)}}}\right\}.$$

2895. (Proposed by J. J. WALKER, M.A.)—At any point on a cusped cubic a tangent is drawn meeting the cubic in a second point; from the first point a line is drawn touching the cubic in a third point. Prove that the lines drawn from the cusp to these points form with the cuspidal tangent a harmonic pencil.

Solution by W. H. H. HUDSON, M.A.



Let the tangent at A meet the cubic in a, and from A draw AB tangent to the cubic. Let C be the cusp. CT the tangent thereat. It is required to prove that CB, Ca, CT, CA form an harmonic pencil. Take ABC as triangle of reference. The equation of the cubic is $\gamma(l\alpha+m\beta)^2+n\alpha^2\beta=0$.

[This may be obtained by taking the complete equation of 10 terms, and simplifying it from the considerations (1) that the cubic passes through A, B, C, (2) AB touches at B, (3) any line through C meets it in two points, then (4) the two lines that meet it in three points at C are coincident.]

$l\alpha+m\beta=0$ is the equation of CT,

$l^2\gamma+n\beta=0$ is the equation of the tangent at A (Aa),

and it meets the cubic where $2l\alpha+m\beta=0$, which is therefore the equation of CA. From the form of these equations we see that CB, Ca, CT, CA form an harmonic pencil.

[Mr. THOMSON and the PROPOSER remark that the theorem follows at once from Salmon's *Higher Plane Curves*, Art. 178.]

2902. (Proposed by J. J. WALKER, M.A.)—The circle passing through three points on a parabola, the normals at which co-intersect, always passes through the vertex; and if the point of co-intersection of the normals describe a coaxial conic having the vertex as centre and axes in the ratio of $m:n$, the locus of the centre of the circle will be a conic having the focus as centre, and axes in the ratio of $2m:n$.

Solution by the PROPOSER; W. ROBERTS; R. TUCKER, M.A.; *and others.*

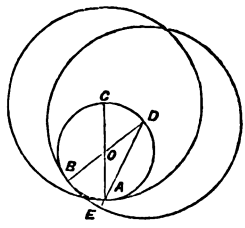
Let (α, β) be the common point of the normals, and $y^2=4ax$ the parabola; then the equation to the circle is readily shown to be

$$x^2+y^2-(\alpha+2a)x-\tfrac{1}{2}\beta y=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

From this equation we see that the circle passes through the vertex; also that it (x', y') be the centre, $2x'=\alpha+2a$, $4y'=\beta$. Let $n^2\alpha+m^2\beta^2=k^2$ be the locus of (α, β), then substituting $2(x'-a)$ for α and $4y'$ for β, $n^2x'^2+4m^2y'^2=\frac{1}{4}k^2$ is the locus of (x', y')—a conic having its centre at $(-a, 0)$ and axes in the ratio of $2m:n$.

2898. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—A circle rolls with internal contact on a fixed circle of half its radius; prove that the envelope of any chord of the rolling circle is a circle which reduces to a point for a diameter.

Solution by W. H. H. HUDSON, M.A.; F. D. THOMSON, M.A.; R. TUCKER, M.A.; *and others.*



Let the diameter considered be the line joining the centres when the point of contact is A; when B is the point of contact let D be the centre of the large circle. Join DA; and produce it to meet the large circle in E.

Since BOA = 2ODA, arc BA = arc BE; therefore E is the point to which A has rolled, and DE is the new position of AC. Hence the diameter always passes through A.

Now conceive a chord parallel to this diameter. It is always at a constant distance from it. Hence, if from A we draw a perpendicular upon it, this perpendicular is constant. It therefore envelopes a circle with centre A, which degenerates to the point A when the chord becomes a diameter.

2875. (Proposed by the REV. J. WOLSTENHOLME, M.A.)—Any tangent to a conic is, of course, divided in involution by three other tangents, and the lines joining their points of intersection to one of the foci of the conic; prove that the distance between the double points of the involution subtends a right angle at the focus of the conic. The locus of these double points, the three tangents being fixed, is a cubic having a double point at the focus, whose nature I have not examined.

Solution by ARCHER STANLEY.

Let A, B, C be fixed tangents to the given conic, and X a variable one; moreover, let P and P′ be the tangents from a given point p. (In the question p is a focus, P and P′ therefore pass through the circular points at infinity, and all harmonic conjugates relative to them are orthogonal.) Now, by a well known theorem, the rays from p to the three pairs of opposite intersections of the quadrilateral ABCX are in involution, and to this involution likewise belong all pairs of tangents from p to the several conics inscribed in the same quadrilateral. Now the given conic obviously belongs to this series, hence P, P′ form a pair of conjugate rays of the involution; the double rays D_1, D_2 thereof must, consequently, be harmonic conjugates relative to P and P′, no matter what position X may have. This proves the first part of the theorem.

With respect to the locus of the intersections of X with the double rays D_1 and D_2, I observe that the latter are the tangents at p to the two conics inscribed in ABCX which pass through p. Hence, not only does X determine the harmonic conjugates D_1, D_2 relative to P and P′, but, conversely, every

such line-pair D_1, D_2 determines its corresponding tangent X; the latter is, in fact, the fourth tangent common to the given conic, and to that which is inscribed in ABC so as to touch D_1 (or D_2) at p. It is easy, therefore, to determine the order of the locus, for through any point ξ of an arbitrary line L will pass one line-pair (D_1, D_2) corresponding to one tangent X, which cuts D say in x. Again, two tangents X pass through every point x; and to them will correspond two line-pairs (D_1, D_2) cutting L in four points ξ. Hence, between the points ξ and x there is a (1, 4) correspondence, and consequently there are five points in which x and ξ are coincident. It is easy to see, however, that one of the latter is always on P and another on P'; deducting these, which are not proper points of the required locus, we conclude that the latter is a cubic, of which p is a double point and P, P' the tangents thereat, since on every line through p there is but one point of the locus which is not coincident therewith, and on P (and P') there are none. (When p is a focus of the given conic, it is also a conjugate point on the cubic.) It is easy to find the nine intersections of the cubic with the fixed tangents A, B, C. Three of them will be the points a, b, c, where the sides A, B, C are intersected by the connector of p with the respective opposite vertices (BC), (CA), (AB); for instance, a is obviously a double point of the involution on the second tangent X which passes through it. The remaining two points of the cubic on A may be best found by conceiving X to coincide with it, or, in other words, to intersect it in its point of contact α; for then a and α will be one pair of conjugate points of the involution on A, another pair will be the intersections (AB), (AC). The points where A cuts the cubic, therefore, will be those which divide both the segments $\overline{a\alpha}$ and $\overline{(AB)\,(AC)}$ harmonically. In this manner the *order* of the locus might have been more readily determined.

2003. (Proposed by W. H. LAVERTY.)—If A', B', C', D', E', F' be the minors which are the coefficients of A, B, C, D, E, F, respectively, in the different expansions of

$$\begin{vmatrix} A & F & E \\ F & B & D \\ E & D & C \end{vmatrix};$$ and if ϕ_1 denote either $(A', B', C', D', E', F')(\lambda, \mu, \nu)^2$, or $(A', B', C', D', E', F')(a, b, c)^2$;

if $\phi_2 = (1, 1, 1, -\cos A, -\cos B, -\cos C)(\lambda, \mu, \nu)$;

if also $P = \left(\frac{d\phi_2}{d\mu}\right)\left(\frac{d\phi_1}{d\nu}\right) - \left(\frac{d\phi_1}{d\mu}\right)\left(\frac{d\phi_2}{d\nu}\right)$; and Q and R denote similar quantities involving ν, λ, and λ, μ, respectively; then show that, if a focus of the conic $(A, B, C, D, E, F)(\alpha, \beta, \gamma)^2 = 0$ be known to be in the line $\lambda\alpha + \mu\beta + \nu\gamma = 0$, its coordinates (λ', μ', ν') are proportional to the minors of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ \lambda & \mu & \nu \\ P & Q & R \end{vmatrix}$, and the equation to the diameter of the conic is

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \lambda' & \mu' & \nu' \\ \left(\frac{d\phi_1}{da}\right) & \left(\frac{d\phi_1}{db}\right) & \left(\frac{d\phi_1}{dc}\right) \end{vmatrix} = 0.$$

Solution by the PROPOSER.

It is evident that if we find the pole of $\lambda\alpha+\mu\beta+\nu\gamma=0$, and from this pole draw a perpendicular to the line, this perpendicular will meet the line in the focus. The coordinates, then, of the pole of the line are

$$\frac{\alpha'}{\left(\frac{d\phi_1}{d\lambda}\right)}=\frac{\beta'}{\left(\frac{d\phi_1}{d\mu}\right)}=\frac{\gamma'}{\left(\frac{d\phi_1}{d\nu}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and the condition that a line $L\alpha+M\beta+N\gamma=0$ ..... (2)

may be perpendicular to $\lambda\alpha+\mu\beta+\nu\gamma=0$, we should find to be

$$L(\lambda-\mu\cos C-\nu\cos B)-M(\mu-\nu\cos A-\lambda\cos C)+N(\nu-\lambda\cos B-\mu\cos A)=0,$$

or $$L\left(\frac{d\phi_2}{d\lambda}\right)+M\left(\frac{d\phi_2}{d\mu}\right)+N\left(\frac{d\phi_2}{d\nu}\right)=0 \quad \ldots\ldots\ldots\ldots (3).$$

Also we have, from (1), since $(\alpha'\ \beta'\ \gamma')$ is on the line,

$$L\left(\frac{d\phi_1}{d\lambda}\right)+M\left(\frac{d\phi_1}{d\mu}\right)+N\left(\frac{d\phi_1}{d\nu}\right)=0 \quad \ldots\ldots\ldots\ldots (4);$$

therefore, from (2), (3), (4), we get (by eliminating L, M, N) for the equation of the perpendicular $P\alpha+Q\beta+R\gamma=0$ ..... (5),

and for the intersection of this with $\lambda\alpha+\mu\beta+\nu\gamma=0$ we have (for the coordinates of the focus)

$$\frac{\lambda'}{\begin{vmatrix}\mu & \nu\\ Q & R\end{vmatrix}}=\frac{\mu'}{\begin{vmatrix}\nu & \lambda\\ R & P\end{vmatrix}}=\frac{\nu'}{\begin{vmatrix}\lambda & \mu\\ P & Q\end{vmatrix}} \quad \ldots\ldots\ldots\ldots (6);$$

and since the coordinates of the centre are

$$\frac{\bar{\alpha}}{\left(\frac{d\phi_1}{da}\right)}=\frac{\bar{\beta}}{\left(\frac{d\phi_1}{db}\right)}=\frac{\bar{\gamma}}{\left(\frac{d\phi_1}{dc}\right)} \quad \ldots\ldots\ldots\ldots (7),$$

we have, from (6) and (7), for the equation of the diameter

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \lambda' & \mu' & \nu' \\ \left(\frac{d\phi_1}{da}\right) & \left(\frac{d\phi_1}{db}\right) & \left(\frac{d\phi_1}{dc}\right) \end{vmatrix}=0.$$

2730. (Proposed by J. J. WALKER, M.A.)—If the determinant $D=\begin{vmatrix} a & e & f & g \\ e & b & h & i \\ f & h & c & k \\ g & i & k & d \end{vmatrix}$ vanish, prove that $(cd)=\frac{(ci)(i)+(dh)(h)}{(b)}$, where (b), (i), (h), (cd), ... are minors formed by omitting the rows and columns in which the bracketed constituents stand.

Solution by the PROPOSER.

In the *Messenger of Mathematics*, No. XIII., p. 27, I have shown that $\begin{vmatrix} (cd) & (dh) & (ci) \\ (dh) & (bd) & (bk) \\ (ci) & (bk) & (bc) \end{vmatrix} = a^2D$; consequently, when $D=0$,

$$(cd) = \frac{(bd)(ci)^2+(bc)(dh)^2-2(dh)(ci)(bk)}{(bd)(bc)-(bk)^2}$$

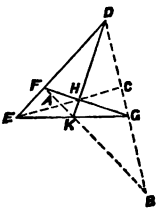
$$= \frac{(ci)\{(ci)(bd)-(bk)(dh)\}+(dh)\{(dh)(bc)-(bk)(ci)\}}{(bd)(bc)-(bk)^2}$$

$$= \frac{(ci)(i)+(dh)(h)}{(b)} \text{ (see p. 25),}$$

which is the type of six similar relations.

2808. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—Four fixed tangents are drawn to a conic S; three other conics are drawn osculating S in any the same point P, and each passing through the ends of a diagonal of the circumscribed quadrilateral; prove that the tangents to these conics at the ends of the diagonal meet in one point P′, and that the locus of P′ is a curve of the sixth degree and fourth class, having two cusps on every diagonal, and touching S at the points of contact of the four tangents.

Solution by F. D. THOMSON, M.A.



With the letters of the figure take the triangle ABC formed by the diagonals as the triangle of reference, and let the equation to the line DEF be $ax+by+cz=0$. Then the coordinates of the points in the figure may be taken as

D $(0, c, -b)$, G $(0, c, b)$,
E $(-c, 0, a)$, H $(c, 0, a)$,
F $(b, -a, 0)$, K $(b, a, 0)$.

By properly determining the constants, we can take as the equation to the given conic

$$x^2+y^2+z^2=0 \quad \ldots\ldots\ldots \text{(i)}.$$

Since DEF touches this, we get the condition

$$a^2+b^2+c^2=0 \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

But, by Salmon's *Conics*, Art. 251, the equation to a conic osculating (i) at the point $(x'y'z')$ is $x^2+y^2+z^2+(xx'+yy'+zz')(lx+my+nz)=0$ .... (iii),

where $lx'+my'+nz'=0$ ..... (iv).

If (iii) pass through the points D and G, we find, substituting the values of the coordinates,

$$m = \frac{-y'(b^2+c^2)}{(cy')^2-(bz')^2} = \frac{a^2y'}{(cy')^2-(bz')^2}, \quad n = \frac{-a^2z'}{(cy')^2-(bz')^2},$$

and therefore $$l = \frac{a(x'^2-y'^2)}{x'\left\{(cy')^2-(bz')^2\right\}};$$

therefore (iii) becomes

$$S \equiv x^2+y^2+z^2+\frac{a^2}{(cy')^2-(bz')^2}\left\{(x'^2-y'^2)\frac{x}{x'}+yy'-zz'\right\} \times \left\{xx'+yy'-zz'\right\} = 0.$$

Now, if (xyz) be the pole of DG, *i. e.* of $x=0$, (xyz) is given by

$$\frac{dS}{dy} = 0, \quad \frac{dS}{dz} = 0.$$

These equations, after reductions by means of (ii) and $x'^2+y'^2+z'^2=0$, become

$$\frac{b^2x'^2}{a^2}y - \frac{y'^3}{x'}x = 0 = \frac{-c^2x'^2}{a^2}z + \frac{z'^3}{z}x,$$

or $$\frac{a^2x}{x'^3} = \frac{b^2y}{y'^3} = \frac{c^2z}{z'^3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (v),$$

which, from its symmetrical form, is evidently the same point as we should have obtained from either of the other two osculating conics which pass through EH and FK. Equations (v) give us

$$\frac{x'^2}{(a^2x)^{\frac{1}{3}}} = \frac{y'^2}{(b^2y)^{\frac{1}{3}}} = \frac{z'^2}{(c^2z)^{\frac{1}{3}}} = \frac{0}{(a^2x)^{\frac{1}{3}}+\&c.};$$

therefore the locus of the point is the curve

$$(ax)^{\frac{1}{3}}+(b^2y)^{\frac{1}{3}}+(c^2z)^{\frac{1}{3}} = 0, \quad \text{or} \quad (a^4x^2+b^4y^2+c^4z^2)^3 = 17a^4b^4c^4x^2y^2z^2,$$

a curve of the sixth order having two cusps on each diagonal, at the points where they are cut by the conic

$$a^4x^2+b^4y^2+c^4z^2 = 0.$$

If we transform to *tangential* coordinates, the equation to the point given by (v) is $$\frac{x'^3}{a^2}\lambda + \frac{y'^3}{b^2}\mu + \frac{z'^3}{c^2}\nu = 0 \ldots\ldots\ldots\ldots (vi);$$

therefore, to find the locus, differentiate, and we have

$$\frac{x'^2}{a^2}\lambda\,dx' + \frac{y'^2}{b^2}\mu\,dy' + \frac{z'^2}{c^2}\nu\,dz' = 0, \quad \text{when} \quad x'\,dx'+y'\,dy'+z'\,dz' = 0;$$

therefore $$\frac{x'}{a^2}\lambda = \frac{y'}{b^2}\mu = \frac{z'}{c^2}\nu;$$

therefore, eliminating x', y', z', $$\frac{a^4}{\lambda^2}+\frac{b^4}{\mu^2}+\frac{c^4}{\nu^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (vii),$$

the tangential equation to the locus. This is of the 4th class, and is satisfied by the lines $(\pm a, \pm b, \pm c)$, and therefore touches the sides of the quadrilateral. Also the pole of the line $(\lambda'\mu'\nu')$ is $\lambda\lambda'(b^4\nu'^2+c^4\mu'^2)+\&c. = 0$; therefore point of contact of (a, b, c), *i. e.* of DEF, is $a\lambda+b\mu+c\nu = 0$, or in trilinears $\frac{x}{a}+\frac{y}{b} = \frac{z}{c}$, which is the point of contact of DEF with the given conic; therefore, &c.

Cor.—If the points F and K be the two real foci of the given conic, and E and H the circular points at infinity, B is the centre and DG the minor axis. Hence the osculating conic through EH and the point $(x'y'z')$ becomes the circle of curvature at $x'y'z'$; that through DG and $(x'y'z')$ an osculating conic through the imaginary foci; and that through FK and $(x'y'z')$ an oscu-

lating conic through the real foci. Hence the locus of the centre of curvature at $(x'y'z')$ is the same as that of the poles of the minor and major axes of the original conic with respect to the other two osculating conics.

2787. (Proposed by C. W. MERRIFIELD, F.R.S.) — The value of $\iiint x^2\,dx\,dy\,dz$ taken over the whole volume of a tetrahedron through one of whose corners the plane of (yz) passes, is $\frac{1}{5}V(a_1^2+b_1^2+c_1^2)$, where V is the volume, and a_1, b_1, c_1 are the distances of the middle points of the opposite edges from the plane of reference.

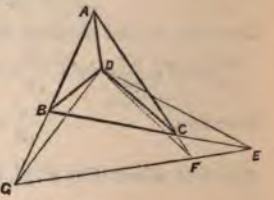
Solution by the PROPOSER.

Let us first find the value of the integral taken over the volume of a pyramid having its *base* in the plane of reference. Let the pyramid have a square base, and let one of its edges be perpendicular to its base. Let its height be h and the side of its base k. The Cartesian equation of one of its edges will be $\frac{x}{h}+\frac{y}{k}=1$, and we shall have to find $\int_0^h x^2y^2\,dx$ subject to that equation. On substituting $k\left(1-\frac{x}{h}\right)$ for y, and performing the integrations, we find its value to be $\frac{1}{30}k^2h^3 = \frac{1}{30}$ base × (altitude)^3. It is evident that, whatever be the shape of the pyramid, we shall always have

$$\iiint x^2\,dx\,dy\,dz = \tfrac{1}{30}\text{ base} \times (\text{altitude})^3,$$

provided the base of the pyramid be in the plane of reference. Moreover, the volume is $\frac{1}{3}$ base × (altitude).

Now let ABCD be the tetrahedron, and DEFG the plane of reference. Produce the four planes of the pyramid to intersect with DEFG, as shown in the figure, and let the distances of A, B, and C from the plane of reference be α, β, and γ respectively.



Since the pyramids ADFG and BDEF intersect so as to have the quadrangular pyramid DBCFG for a common frustum, we have

$$\text{ABCD} = \text{ADFG} + \text{CDEF} - \text{BDEG};$$

and also, if we call MI the integral $\iiint x^2\,dx\,dy\,dz$ taken over the corresponding pyramid, we have

$$\text{MI}.\text{ABCD} = \text{MI}.\text{ADFG} + \text{MI}.\text{CDEF} - \text{MI}.\text{BDEG}.$$

But $$\text{MI}.\text{ADFG} = \frac{\alpha^3}{30}\text{DFG} = \frac{\alpha^2}{10}\text{ADFG},$$

and similarly $$\text{MI}.\text{CDEF} = \frac{\gamma^2}{10}\text{CDEF}, \quad \text{MI}.\text{BDEG} = \frac{\beta^2}{10}\text{BDEG}.$$

Again, $$\text{BDEG} = \text{ADFG} + \text{CDEF} - \text{ABCD};$$

therefore $$10\text{MI}.\text{ABCD} = (\alpha^2-\beta^2)\,\text{ADFG} + (\gamma^2-\beta^2)\,\text{CDEF} + \beta^2.\text{ABCD};$$

next, $AFGD : ABCD = \triangle AFG : \triangle ABC = AF.AG : AC.AB$
$= \alpha.\alpha : (\alpha-\gamma)(\alpha-\beta)$;

also, $CEFD : ABCD = \triangle CEF : \triangle ABC = FC.CE : AC.CB$
$= \gamma.\gamma : (\alpha-\gamma)(\beta-\gamma)$;

$$\therefore \frac{10\,MI.ABCD}{\text{vol. } ABCD} = \frac{\alpha^2(\alpha^2-\beta^2)}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\gamma^2(\gamma^2-\beta^2)}{(\gamma-\alpha)(\gamma-\beta)} + \beta^2$$
$$= \frac{\alpha^2(\alpha+\beta)}{\alpha-\gamma} - \frac{\gamma^2(\gamma+\beta)}{\alpha-\gamma} + \beta^2 = \frac{\alpha^3-\gamma^3+\beta(\alpha^2-\gamma^2)}{\alpha-\gamma} + \beta^2$$
$$= \alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta$$
$$= \tfrac{1}{2}(\alpha+\beta)^2+\tfrac{1}{2}(\beta+\gamma)^2+\tfrac{1}{2}(\gamma+\alpha)^2;$$

therefore $MI.ABCD = \frac{1}{2}V(a_1^2+b_1^2+c_1^2)$.

MI is not a moment of inertia, being taken with reference to a plane instead of a line; but the formula, as it involves the squares, evidently holds for the moment of inertia about an axis. The common moment about the plane is

$$\tfrac{1}{4}V(a_1+b_1+c_1).$$

Moreover, the common moment and moment of inertia of a triangle about a line passing through one of its angular points are, severally, A being the area of the triangle,

$$\tfrac{1}{3}A(a_1+b_1+c_1) \quad \text{and} \quad \tfrac{1}{3}A(a_1^2+b_1^2+c_1^2).$$

The similarity of these forms is very remarkable.

2701. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—In a right conoid whose axis is the axis of z, show that the radii of principal curvature at the point $(r\cos\theta,\ r\sin\theta,\ z)$ are given by the equation

$$4\rho^2\left(\frac{dz}{d\theta}\right)^2 + r\rho\frac{d^2z}{d\theta^2}\left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^{\frac{1}{2}} = \left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^2.$$

I. *Solution by the* Rev. J. L. KITCHIN, M.A.

The radii of principal curvature are given by the equation

$$(r_1t-s^2)\rho^2 - \left\{(1+p^2)t-2pqs+(1+q^2)r_1\right\}\rho(1+p^2+q^2)^{\frac{1}{2}} + (1+p^2+q^2)^2 = 0 \ldots\ldots (1).$$

The general equation of a conoid, axis of surface axis of z, is $z = \phi\left(\frac{y}{x}\right)$; and since $x = r\cos\theta$, $y = r\sin\theta$, this becomes $z = \phi(\tan\theta) = f(\theta)$; therefore z is independent of r. Hence we find

$$\frac{dz}{dx} = -\frac{1}{r}\sin\theta\frac{dz}{d\theta}, \quad \frac{dz}{dy} = \frac{1}{r}\cos\theta\frac{dz}{d\theta};$$

and hence $1+p^2+q^2 = 1+\frac{1}{r^2}\left(\frac{dz}{d\theta}\right)^2 = \frac{1}{r^2}\left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}$; also

$$\frac{d^2z}{dx^2} = -\frac{d}{dx}\left(\frac{1}{r}\sin\theta\frac{dz}{d\theta}\right) = -\frac{d}{dr}\left(\frac{1}{r}\sin\theta\frac{dz}{d\theta}\right)\frac{dr}{dx} - \frac{d}{d\theta}\left(\frac{1}{r}\sin\theta\frac{dz}{d\theta}\right)\frac{d\theta}{dx}.$$

Effecting the differentiations and reducing, remembering that $\frac{dz}{d\theta}$ is a function of θ *only*, we get

$$\frac{d^2z}{dx^2} = \frac{\sin^2\theta}{r^2}\frac{d^2z}{d\theta^2} + \frac{2\sin 2\theta}{r^2}\frac{dz}{d\theta} = r_1;$$

similarly $$\frac{d^2z}{dy^2} = \frac{\cos^2\theta}{r^2}\frac{d^2z}{d\theta^2} - \frac{2\sin 2\theta}{r^2}\frac{dz}{d\theta} = t;$$

and $$\frac{d^2z}{dx\,dy} = -\left(\frac{2\cos 2\theta}{r^2}\frac{dz}{d\theta} + \frac{\sin 2\theta}{2r^2}\frac{d^2z}{d\theta^2}\right);$$ hence

$$r_1 t - s^2 = \frac{1}{r^4}\left\{\frac{\sin^2 2\theta}{4}\frac{d^2z}{d\theta^2} + \sin 4\theta\frac{d^2z}{d\theta^2}\frac{dz}{d\theta} - 4\sin^2 2\theta\left(\frac{dz}{d\theta}\right)^2\right\}$$
$$- \frac{1}{r^4}\left\{4\cos^2 2\theta\left(\frac{dz}{d\theta}\right)^2 + \sin 4\theta\frac{d^2z}{d\theta^2}\frac{dz}{d\theta} + \frac{1}{4}\cos^2 2\theta\left(\frac{d^2z}{d\theta^2}\right)^2\right\}$$
$$= -\frac{4}{r^4}\left(\frac{dz}{d\theta}\right)^2 \ldots\ldots\ldots$$

We find also that $(1+p^2)\,t - 2pqs + (1+q^2)\,r_1 = \frac{1}{r^2}\frac{d^2z}{d\theta^2}$.

Hence (1) becomes

$$-\frac{4}{r^4}\left(\frac{dz}{d\theta}\right)^2\rho^2 - \frac{1}{r^2}\rho\frac{d^2z}{d\theta^2}\frac{\left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^{\frac{1}{2}}}{r} + \frac{\left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^2}{r^4} = 0,$$

or $$4\rho^2\left(\frac{dz}{d\theta}\right) + r\rho\frac{d^2z}{d\theta^2}\left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^{\frac{1}{2}} = \left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^2.$$

II. *Solution by the* Proposer.

Let the increments of r, θ, z, at a point Q near the given point P, be $\lambda\delta\theta$, $\delta\theta$, δz; then the tangent plane at P is the limiting position of the plane containing the generating line $X\sin\theta = Y\cos\theta$, $Z = z$, and passing through an adjacent point of the surface; its equation being taken $X\sin\theta - Y\cos\theta + k(Z-z) = 0$, k is the limit of $\frac{(r+\lambda)\sin\delta\theta}{\delta z}$, or $r \div \frac{dz}{d\theta}$. But if ρ be the radius of curvature of the normal section containing the ultimate position of the line PQ, and p be the perpendicular from Q on the tangent plane at P, 2ρ = limit of $\frac{PQ^2}{p}$.

Now $$PQ^2 = \lambda^2\delta\theta^2 + r^2\delta\theta^2 + \delta z^2,$$

and $$p = \frac{-(r+\lambda\delta\theta)\frac{dz}{d\theta}\sin\delta\theta + r\left(\frac{dz}{d\theta}\delta\theta + \frac{d^2z}{d\theta^2}\frac{\delta\theta^2}{z} + \ldots\right)}{\left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^{\frac{1}{2}}}$$

whence $$\rho = \left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^{\frac{1}{2}}\frac{r^2+\left(\frac{dz}{d\theta}\right)^2+\lambda^2}{r\frac{d^2z}{d\theta^2} - \lambda\frac{dz}{d\theta}};$$

and the principal radii of curvature are the maximum and minimum values of this for different values of λ, and are therefore given by the equation

$$4\rho^2\left(\frac{dz}{d\theta}\right)^2 + r\rho\frac{d^2z}{d\theta^2}\left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^{\frac{1}{2}} = \left\{r^2+\left(\frac{dz}{d\theta}\right)^2\right\}^2.$$

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CORRIGENDA.

Vol. XI.

Page vi., *insert* Question 2003 ..... p. 106.
" viii., *omit* Question 2603 ..... p. 105.
" x., line 4, *omit* the factor 4.
" x., *insert* Question 2730 ..... p. 107.
" xiii., *omit* Question 2782 ..... p. 107.
" xv., Question 2875, *for* p. 106 *read* p. 105.
" 25, line 17, should be as follows:—

$$\frac{3\pi}{a b^3}\int_0^{\frac{1}{2}a} (b\beta^2-\beta^3)\,d\alpha = \frac{6\pi}{b^5}\int_b^{\frac{1}{2}b} (b\beta^2-\beta^3)\,(b-2\beta)\,d\beta$$

" 58, in the Figure, the dotted curve should *touch* the parabola at C.
" 81, line 21, *for* sin b sin c *read* bc.
" 94, line 20, *for* ellipse and a concentric hyperbola, &c., *read* a rectangular hyperbola and a circle whose diameter is any chord of the hyperbola (and therefore the complementary common chord a diameter of the hyperbola).
" 97, line 16, *for* them in 4-plets *read* four in m-plets.
" 109, line 7 from bottom, *for* $\frac{x}{a}+\frac{y}{b}$ *read* $\frac{x}{x}=\frac{y}{b}$.
" 111, line 22, *omit* the factor 4.
" 112, lines 1, 2, *for* $2\sin 2\theta$ *read* $\sin 2\theta$; line 3, *for* $2\cos 2\theta$ *read* $\cos 2\theta$; line 4, *for* $\sin 4\theta$ *read* $\frac{1}{2}\sin 4\theta$, and *for* $4\sin^2 2\theta$ *read* $\sin^2 2\theta$; line 5, *for* $4\cos^2 2\theta$ *read* $\cos^2 2\theta$, and *for* $\frac{1}{4}\cos^2 2\theta$ *read* $\frac{1}{4}\sin^2 2\theta$; lines 6, 9, 25, *omit* the factor 4; line 10, *for* $4\rho^2\left(\frac{dz}{d\theta}\right)$ *read* $\rho^2\left(\frac{dz}{d\theta}\right)^2$; line 2, *for* $\frac{\delta\theta^2}{z}$ *read* $\frac{\delta\theta^2}{2}$; line 22, *for* $-\lambda$ *read* -2λ.

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" 52, line 11, *for* $\frac{1}{ar^2}$ *read* $\frac{2}{ar^2}$; line 12, *omit* the $-$ before $\int$; line 13, *for* $\frac{1}{3ar^2}$ *read* $\frac{2}{3ar^2}$.
" 58, line 25, *for* p. 28 *read* p. 30.
" 71, line 7, *for* (1) *read* (2), and *for* (2) *read* (3).

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2283.	The eccentric angles of three points P, Q, R on an ellipse being α, β, γ, find the relations these quantities satisfy when PQ, QR are (i.) normals at P, Q, (ii.) normals at P, R; and find the area of PQR in the two cases; also the maximum triangles.	36
2296.	If a triangle be formed with the centre of a circle as vertex, and the line joining two points taken at random within the circle as base; show that the chances of the angle at the centre being the greatest, intermediate, or least angle of the triangle, are respectively as $4\pi+6\sqrt{3} : 4\pi-3\sqrt{3} : 4\pi-3\sqrt{3}$	70
2342.	Construct the maximum triangle of given species whose three sides shall touch three given circles.	78
2389.	Deux droites qui divisent harmoniquement les trois diagonales d'un quadrilatère rencontrent en quatre points harmoniques toute conique inscrite dans le quadrilatère.	50
2398.	A point P is laid down in a plane by taking random values for AP and BP, its distances from two fixed points A, B; and the plane is covered with an infinite multitude of such points. Show that their distribution will be such that the circle on AB as diameter will be the line of maximum density; and that all circles through A, B are lines of equal density, the density being infinitely small along the line AB.	33
2399.	Find a point on a sphere such that the triangle determined by the middle points of the three arcs connecting it with three given points on the sphere will have two of its sides given.	58
2410.	If a polygon A, B, C, D, ... M be inscribed in a circle, which is supposed to roll on the inside of a circle of double its radius, prove that from the centre o of the greater circle a pencil of m	

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lines *oa*, *ob*, *oc*, *od*, ... *om* can be drawn such that, for any determined position taken by the given polygon, an infinite number of polygons can be drawn homothetical to it, and having their summits $\alpha, \beta, \gamma, \delta, \ldots \mu$ upon the lines *oa*, *ob*, *oc*, *od*, ... *om*, respectively. 89

2446. PQ is a chord of a conic, equally inclined to the axis with the tangent at P. Any circle through PQ cuts the conic in RS. Show that the harmonic conjugate of RS relative to P lies on the straight line joining Q to the other extremity of the diameter through P. Hence show by inversion that if chords be drawn to a circular cubic through the point where the asymptote cuts the curve, the locus of their middle points is a circle through the double point. 90

2463. It is required to divide any rational function of n into two parts, such that the one and the reciprocal of the other are similar functions of n and of $n+1$ respectively. 78

2526. Through the foci of an ellipse two chords are drawn, and with them as diagonals a quadrilateral is formed; find (1) the maximum or minimum figure when they intersect at a constant angle, and (2) the locus of their point of intersection when the quadrilateral is a maximum or a minimum. 35

2542. A given conic is continually touched by a similar and similarly placed conic; find the locus of a point rigidly connected with the moving conic. 23

2546. Eliminate θ between

$$\sin\theta\left\{x\sin(60^\circ-A-\theta)+y\cos(60^\circ-A-\theta)\right\}+y\sin A=0,$$

$$\sin(A+\theta)\left\{y\cos(60^\circ+\theta)-(x-a)\sin(60^\circ+\theta)\right\}=y\sin A. \ldots 34$$

2549. Solve the simultaneous equations

$$yz+x=14, \quad zx+y=11, \quad xy+z=10. \ldots 40$$

2582. Supposing that two quadrics have double contact, their axes remaining parallel; then (1) determine the *complete* locus of the centre of one quadric relative to the other; and (2) find the surfaces traced out by their chords of contact. 32

2603. In a plane, (1) on every line there are two points harmonic conjugates to the conics through four points ABCD. They are real, if the product of the four triangles ABC, BCD, CDA, DAB, and the perpendiculars from the points ABCD on the line, is positive. If a line does not cut a conic, why are the conjugates on the line real, to any set of four points on the conic? Give a rule for determining the nature of the points on a line, when the product of the triangles is (1) positive, (2) negative.

(2.) Through any point there are two lines conjugate to the conics inscribed in four lines. They are real in those regions of the plane formed by the four lines in which the product of perpendiculars from the point has the same sign as that of points in the convex quadrilateral. Hence every point on or within a conic inscribed in the four lines lies in these regions...30, 57

2608. Show that if (a_1, a_2) are points of contact of tangents from a point A on a cubic curve, and (b_1, b_2) similar points of contact

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of tangents from B, another point on the same curve; and if (a_1, b_1) and (a_2, b_2) respectively are collinear with the point of contact of a tangent from the point where the line AB meets the curve again, then AB $a_1a_2b_1b_2$ lie in the same conic. 21

2615. If $x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0$

be an algebraical equation whose roots are $\alpha, \beta, \gamma, \ldots$, find the equation whose roots are the different values of

$$\frac{dp_1}{da} + \frac{dp_2}{da} + \ldots\ldots + \frac{dp}{da} \quad \ldots\ldots 51$$

2617. Solve the differential equation $x^2 + y^2 + \frac{2(1+p^2)}{q}(y - xp) = k^2$,

where $p = \frac{dy}{dx}$, $q = \frac{d^2y}{dx^2}$, and k is a constant. 103

2623. If a conic pass through the four points of contact of tangents to a cubic from a point (A) on the curve, and through two other points (B, C) on the cubic; then A is the pole of BC with regard to the conic. 53

2668. If the complete solution of the differential equation

$$\frac{d^ny}{dx^n} + f(x)\frac{d^{n-1}y}{dx^{n-1}} + \mathrm{F}\left(\frac{d^{n-2}y}{dx^{n-2}} \ldots \frac{dy}{dx}, y, x\right) = 0$$

be $y = \phi(x, x_1, x_2, \ldots x_n)$, where x_1, x_2, &c. are the n arbitrary constants introduced by integration; prove that

$$\begin{vmatrix} \frac{d\phi}{dx_1} & \frac{d\phi}{dx_2} & \ldots & \frac{d\phi}{dx_n} \\ \vdots & \vdots & & \vdots \\ \frac{d\phi^{(n)}}{dx_1} & \frac{d\phi^{(n)}}{dx_2} & \ldots & \frac{d\phi^{(n)}}{dx_n} \end{vmatrix} e^{\int f(x)\,dx} = \text{constant},$$

where $\phi^{(r)} = \frac{d^r\phi}{dx^r}$ 110

2684. From a point S two straight lines are drawn meeting a circle in A and B, so that the rectangle SA . SB is constant; show that the envelope of the secant AB is a central conic of which S and its conjugate point are foci; and that the conic is an ellipse or hyperbola according as the given rectangle is greater or less than the rectangle contained by the intercepts made by the circle on any straight line through S. 26

2692. Demonstrate by means of a spherical triangle that the same relation between the amplitudes ϕ, ψ, σ, which gives $\mathrm{F}(\phi) + \mathrm{F}(\psi) - \mathrm{F}(\sigma) = 0$ gives also

$$\mathrm{E}(\phi) + \mathrm{E}(\psi) - \mathrm{E}(\sigma) = \kappa^2 \sin\phi \sin\psi \sin\sigma,$$

when κ is the modulus of the elliptic functions. 76

2716. If a circular wheel roll against a vertical wall, show that the curve traced on the wheel by a fixed point in the wall is $\theta = \sin^{-1}\frac{d}{\rho} + \frac{(\rho^2 - d^2)^{\frac{1}{2}}}{a}$, where d is the difference between the radius (a) of the circle and the perpendicular height of the fixed point from the ground. 76

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2734. Prove that

$$n^{n+1} = (n+1)\,S_n - \frac{(n+1)\,n}{1.2}\,S_{n-1} + \frac{(n+1)\,n\,(n-1)}{1.2.3}\,S_{n-2} - \ldots \pm S_0,$$

where S_1, S_2, S_3, ... S_n are the sums of the first, second, third, &c. powers of the terms of the series 1, 2, 3, 4, ... to n terms... 111

2735. The escribed radii of a triangle are in harmonical progression, and the mean escribed radius is a geometrical mean between the inscribed and circumscribed radii: show that the cosine of the mean angle of the triangle is $\frac{8}{9}$, and that the circumscribed radius is nine times the inscribed radius. 31

2742. Prove that if tangents be drawn to a cubic from the real points of inflexion, the 12 points of contact lie by *threes* on 19 straight lines..... 98

2744. A circle passes through the real foci of a conic, prove that the points where the common tangents touch the circle lie on one of the tangents to the conic at the ends of the minor axis. 108

2747. Prove that the locus of the foot of the bifocal perpendicular on the tangent plane to a quadric is a concentric sphere whose radius is the primary axis. 27

2749. Let D, E, F be the points of contact of the inscribed conic with the triangle of reference, and let AD cut the conic in P_1. Join BP_1, CP_1, meeting the opposite sides in E_1, F_1, and let now D, E_1, F_1 be the points of contact of a second inscribed conic, cutting AD in P_2. Continue this process; then will the equations of the tangent to the nth conic at P_n and of the line $E_n F_n$ be, respectively,

$$l\alpha - \tfrac{1}{2}(4)^n(m\beta + n\gamma) = 0, \quad \text{and} \quad l\alpha - (4)^n(m\beta + n\gamma) = 0. \ldots\ldots 88$$

2767. If $1 + a_1 r + a_2 r^2 + a_3 r^3 + \ldots$ represents a series of which a_1, a_2, a_3, &c., are the 2nd, 3rd, 4th, &c., terms of the pth order of figurate numbers; show that the sum to n terms of this series is $\dfrac{1 - r^n r_p^{-n}}{(1-r)^p}$, where r_p^{-n} denotes the sum of the first p terms in the expansion of r^{-n}. 97

2768. Let $v^3 - av^2 + bv - c = 0$ be one of the five cubics whose roots are simultaneous values of x, y, z, satisfying the three equations in Question 2549 [viz., $yz + x = 14$, $zx + y = 11$, $xy + z = 10$; see *Reprint*, Vol. IX., p. 66, and Vol. XII., p. 40]; show that b must be one of the five roots of the quintic

$$\beta^5 - 140\beta^4 + 7608\beta^3 - 202044\beta^2 + 2634224\beta - 13531024 = 0,$$

and that the corresponding values of a, c are given by the equations $a + b = 35$, $(32 - b)\,c + (36 - b)\,b = 404$; also that the five values of c are the roots of the quintic

$$\gamma^5 + 4\gamma^4 - 2368\gamma^3 + 61452\gamma^2 - 572256\gamma + 1783296 = 0 \ldots\ldots 43$$

2772. The curves

$$y^2 = 4ax \ldots (1), \quad y = a\,(\epsilon^{\frac{x}{2a}} + \epsilon^{-\frac{x}{2a}}) \ldots (2), \quad \text{and} \quad \sin\frac{y}{2a} = \epsilon^{\frac{x}{2x}} \ldots (3),$$

are referred to the same origin and axes; show that, at the

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points for which a tangent (2) is at right angles to tangents to (1) and (3), the radii of curvature are in continued proportion... 33

2782. 1. A given point is known to be within a certain circle of given radius, but unknown position; find the chance that another given point is also within the circle.

2. Three given points are known to be within a certain circle, which is otherwise altogether unknown; determine the most probable position of its centre.

3. Two given points are known to be within a circle, and a third given point is known to be outside it; determine the most probable position of its centre. 28

2783. A series of n letters consists of m_1 groups each containing p_1 letters, m_2 groups each containing p_2 letters, &c.; show that the number of combinations of the n letters taken r at a time, under the restriction that no two letters of the same group shall enter into the same combination, is

$$\Sigma \frac{\lfloor m_1}{\lfloor \alpha \lfloor m_1-\alpha} \cdot \frac{\lfloor m_2}{\lfloor \beta \lfloor m_2-\beta} \ldots\ldots p_1^{\alpha} p_2^{\beta} \ldots\ldots,$$

where α, β, ... &c. take all positive integral values, less than m_1, m_2, ... respectively, consistent with the relation $\alpha+\beta+\gamma+\ldots = r$. 49

2786. If ρ be the angular radius of a primary or secondary rainbow, corresponding to angles (θ, θ') of incidence and reflexion at the surface of the rain-drop, and n be the number of internal reflexions, prove that $\cos \frac{1}{2}(n\pi-\rho) = \cos(\frac{1}{2}n\pi-\theta)(n \sin\theta')^{n+1}$. ... 109

2790. Find the mean distance of all the points in a right circular cylinder from one end of the axis 52

2793. C is the single focus of a semicubical parabola, and from any point O three tangents are drawn to the curve; if CD, CE, CF be perpendicular to them, show that DE and CF are equally inclined to the direction of the infinite branches. 21

2801. Show that the number of pairs of numbers which have a given number G for their G. C. M., and another given number L (of course a multiple of G) for their L. C. M., is 2^{n-1}, where n is the number of prime bases the product of whose powers is equal to $\left(\frac{L}{G}\right)$. 22

2802. Prove that out of a set of numbers consisting of the first n coefficients in $(1+x)^{2n-1}$, or of the first n coefficients and the half of the middle coefficient in $(1+x)^{2n}$, terms may be selected, taken some positively and others negatively, so as to produce the sum 2^{n-1}. For example, if $n = 6$, out of the set 1, 11, 55, 165, 330, 462 we may extract

$$-462+330+165-1 = 32,$$

and out of the set 1, 12, 66, 220, 495, 792, $\frac{1}{2}$(924) we may extract $495-\frac{1}{2}(924)-1 = 32$ 17

2803. Through every point A on a conic pass three circles which osculate the conic elsewhere, say in B, C, D. Prove that A, B, C, D lie on the circumference of a circle, and find the envelope of the latter. 37, 90

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2806. L, M, N are the real points of inflexion of a cubic; PQR is the triangle formed by the tangents at L, M, N; p, q, r are the points forming with L, M, N harmonic sections on the sides of PQR. Prove that the Hessian of the cubic touches the triangle PQR at the points p, q, r. Also if P'Q'R' be the triangle formed by the tangents to the Hessian at L, M, N, then PP', QQ', RR' meet in the point of intersection of the harmonic polars of L, M, N. 63

2811. If

$$u = \sum_{k=0}^{k=p-2} \frac{\Pi(m+n)r}{\Pi\{mr+k(p-1)\}\,\Pi\{nr-k(p-1)\}} \equiv c \pmod{p},$$

where p is any odd prime number; m, n, and r any positive integers; and k takes all integral values between $\frac{mr}{p-1}$ and $-\frac{mr}{p-1}$, inclusive of those quantities, if integral; show that c has the same value for all values of r which have the same G. C. M. with $(p-1)$. 29

2818. Given a circle S and a straight line α not meeting S in real points; O, O' are the two-point circles to which, and S, α is the radical axis; two conics are drawn osculating S in the same point P, and having one focus at O, O' respectively; prove that the corresponding directrices coincide. 65

2820. Find the locus of the intersection of perpendicular normals of a lemniscate. 47

2821. Show that the mean value of the distance from one of the foci of all points within a given prolate spheroid is $\frac{1}{4}a(3+e^2)$, $2a$ being the axis and e the eccentricity. 50

2829. Through the fourth point of intersection of two conics circumscribing the triangle of reference (whose equations are $l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$, $l'\beta\gamma + m'\gamma\alpha + n'\alpha\beta = 0$) is drawn a straight line meeting the curves again in P, P': prove that the tangents at P, P' intersect on the tricusped quartic

$$\left\{\beta\gamma\left(\frac{1}{mn'}-\frac{1}{m'n}\right)\right\}^{\frac{1}{3}} + \left\{\gamma\alpha\left(\frac{1}{nl'}-\frac{1}{n'l}\right)\right\}^{\frac{1}{3}} + \left\{\alpha\beta\left(\frac{1}{lm'}-\frac{1}{l'm}\right)\right\}^{\frac{1}{3}} = 0. \quad 20$$

2833. (1.) Through every point A on a central conic pass three circles, which osculate the conic elsewhere, say in B, C, D. Prove that the diameter of the circle passing through A, B, C, D is $= \frac{a^2+3b^2}{2b}$ when greatest, and $\frac{b^2+3a^2}{2b}$ when least.

(2.) Prove also that the distance between the centre of this circle, and the centre of the circle osculating the conic at B, C, or D, is when greatest $= \frac{3}{4}\left(\frac{a^2+b^2}{b}\right)$, and when least $= \frac{3}{4}\left(\frac{a^2-b^2}{a}\right)$, where a and b are the semi-axes of the conic. ... 37

2840. A circle is described through the foci of a conic, and from a point P on it tangents PA, PB are drawn to the curve, meeting the circle again in Q and R. Show that the other tangents to the conic from Q and R meet on the circle. 24

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2841. A given cylindrical vessel, filled with water, is placed with its base upon a horizontal plane. It is required to find the angle of inclination to which the plane must be raised before the vessel will fall, the water being at liberty to overflow the top. The base is supposed to be fixed so as to prevent it from sliding, but not from tilting, when the plane is inclined. 19

2843. Two parabolas are described through the points A, B, C, D of Question 2803, show that their axes intersect on a similar concentric conic, and that their envelopes and the loci of their vertices and foci are quartic curves. ..... 37 & 44

2844. To find three positive integral numbers, whose sum, and also the sum of any two of them, shall be a rational cube. 79

2849. An ellipse is described about a triangle having the *centroid* for its centre; show that straight lines from any point on it parallel to the sides cut the sides in two sets of three points which are collinear. 23

2852. The four points being as in 2803, prove that (1) If a circle be described concentric with the conic, and bisecting the intervals on the axis between the foci, the locus of the centre of the circle ABCD is the conic polar-reciprocal to the given one with respect to the above circle. (2) That the circle ABCD meets the chord common to the conic, and the osculating circle at A, in points equidistant from the centre of the conic. (3) That the normal at B, and the line joining the centre of the osculating circle at B with the centre of the circle ABCD, are equally inclined to the axis of the conic..... 39

2864. A body moves in an ellipse with *uniform* velocity under the attraction of two central forces at the foci. Show that the forces to each focus are always equal, and vary inversely as the product of the focal distances. 29

2867. Being given four circles, show (1) that their twelve centres of similitude are the points of contact of tangents to a cubic curve drawn from three collinear points upon it; (2) that the curve passes also through the intersections of the straight lines joining the centres of the circles two and two together; and (3) that these results constitute a generalization of the theorem, that, four points being given, the intersections of the straight lines joining them two and two, and the bisecting points of those lines limited by the given points, lie upon the same conic. 41

2869. 1. The square of the chord common to an ellipse and its circle of curvature at any point is equal to $\frac{16}{(a^2-b^2)^2}(a^2-r'^2)(r'^2-b^2)r'^2$, where a, b are the semi-axes, and r' the semi-diameter parallel to the tangent at the point.

2. When the chord is a maximum, $3r'^2=a^2+b^2+(a^4-a^2b^2+b^4)^{\frac{1}{2}}$.

3. In the parabola, the square of the common chord is equal to $16p'x$, p' being the parameter, and x the abscissa of the point of contact. 67

2873. 1. Determine the point D in any quadrant of an ellipse, such

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that if the osculating circle at it meet the ellipse again in C, that at C will pass through D.

2. Show that A, B, the other two points on the ellipse, the osculating circles at which also pass through D, are the ends of two semi-diameters, each conjugate to that semi-diameter lying in D's quadrant which is equal to the other. 66

2880. Given the focus, directrix, and length between the extremities of the major and minor axes of an ellipse; show how to find the major axis and the directions of conjugate diameters by an easy geometrical construction. 109

2884. Subtract each of the seven Dominical Letters A, B, C, D, E, F, G from the sum of the other six of them, and then find the coefficient of the term ABCDEFG in the continued product of the seven remainders thus obtained..... 18

2888. 1. If a Cartesian oval has two imaginary axial foci, and consequently two real extra-axial foci, the tangent at any point bisects (internally or externally) the angle between the vector drawn from the real axial focus and the radius of a circle through the extra-axial foci and the point of contact, the radius being drawn thereto.

2. A conic section may be considered as representing a Cartesian oval either with three real axial foci or with two imaginary and one real axial foci. Regarded in the last point of view, what is the property of the conic corresponding to the general case? 94

2889. The particles of a thin hemispherical shell being supposed to attract with a force varying directly as the distance, prove that the resultant attraction on any point of the rim is the same as if all the matter of the shell were concentrated at the middle point of its axis. 66

2894. An ellipse and hyperbola are confocal, and the asymptotes of the latter lie on the equi-conjugates of the former; prove that any ellipse which passes through the ends of the axes of the given ellipse will cut the hyperbola orthogonally. 27

2900. The foci of the conic $ax^2+2bxy+cy^2+2dx+2cy+f=0$ are given by the intersections of two rectangular hyperbolas concentric with the original conic, whose equations are

$$(b^2-ac)(x^2-y^2)+2(bc-cd)x+2(ac-bd)y+(a-c)f-(d^2-c^2)=0,$$

and $$(b^2-ac)xy+(bd-ae)x+(be-cd)y+bf-de=0. \ldots\ldots 46$$

2907. Given that three events A, B, C occurred in a certain century, A having preceded B, and B having preceded C, this being all the information obtainable; show that the probability (p) that the date of the event B was not distant more than n years from the middle of the century is

$$p=3\frac{n}{100}-4\left(\frac{n}{100}\right)^3. \ldots\ldots 55$$

2909. In a bicircular quartic, the points of contact of the four single tangents drawn from the centre of a circle in which four foci, real or imaginary, lie, are on the circle, and the corresponding points of contact of double tangents also lie on a circle. 87

2910. A circle (of radius r) is described with its centre on the minor

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axis of a given ellipse (of eccentricity e) at a distance from the centre of the ellipse equal to er: prove that the tangents to this circle at any points where it meets the ellipse will touch the minor auxiliary circle. 72

2913. Straight lines are drawn from the angles A, B, C of the triangle ABC, through a point O within it, meeting the opposite sides in D, E, F; show that the locus of O, when the perpendiculars from A, B, C upon EF, FD, DE meet in a point, is a cubic through A, B, C. Find also the locus of the latter point. 60

2914. If ABC be a triangle inscribed in and concentroidal with, an ellipse whose semi-axes are a, b; prove that (1) the sum of the squares of the sides of such a triangle is constant, and equal to $\frac{8}{3}(a^2+b^2)$; (2) the sum of the squares of the lines joining A, B, C with the fourth point D, in which the circumscribing circle cuts the ellipse, is equal to $3\left\{\frac{1}{3}(a^2+b^2)+r^2\right\}$, r being the semi-diameter drawn to the point; and (3) the sum of the squares of the diameters of the ellipse passing through the points A, B, C, is equal to the sum of the squares of three conjugate diameters. 59

2917. If a conic pass through two given points and touch a given conic at a given point, the chord of intersection with the given conic will pass through a fixed point on the given straight line. 54

2920. Imagine a tetrahedron BB′CC′ in which the opposite sides BB′, CC′ are at right angles to each other and to the line joining their middle points M, N; and in which moreover $\overline{CN}^2+\overline{NM}^2+\overline{MB}^2=0$, (or, what is the same thing, the sides CB, CB′, C′B, C′B′ are each $=0$; the tetrahedron is of course imaginary; viz., the lines CC′, BB′ and points M, N may be real; but the distances MB = MB′ and NC = NC′ may be one real and the other imaginary, or both imaginary, but they cannot be both real); the points B, B′ and C, C′ are said to be "skew antipoints." Then it is required to prove that

1. A given system of skew antipoints may be taken to be the nodes (conical points) of a tetranodal cubic surface, passing through the circle at infinity, and which is in fact a Parabolic Cyclide.

2. The equation of the surface may be expressed in the form

$$x(x+\beta)(x+\gamma)+(x+\beta)y^2+(x+\gamma)z^2=0.$$

3. The section through either of the lines ($y=0$, $x+\gamma=0$) and ($z=0$, $x+\beta=0$) is made up of this line and a circle; the two systems of circles being the curves of curvature of the surface; it is required to verify this *à posteriori;* viz., by means of the equation of the surface to transform the differential equation of the curves of curvature in such manner that the transformed equation shall have the integrals

$$y=C(x+\gamma),\quad z=C'(x+\beta). \text{ ..... } 69$$

2922. If $x_1=\frac{x_0}{1+a_0x_0}$, $x_2=\frac{x_1}{1+a_1x_1}$, $x_3=\frac{x_2}{1+a_2x_2}$, &c.;

prove that $x_n=\frac{x_0}{1+hx_0}$, where $h=a_0+a_1+\ldots a_{n-1}$. 51

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2923. In a bicircular quartic, the points of contact of the four single tangents drawn from the centre of a circle on which four foci lie, are on the circle, and the corresponding points of contact of double tangents also lie on a circle..... 56

2924. On a focal chord PSQ of a parabola are taken p, q, on opposite sides of S, such that $Sp \cdot Sq = SP \cdot SQ$, and any parabola is described through p, q, and having its axis parallel to that of the former: prove that their chord of intersection will pass through S. 62

2928. Prove that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} = n - \frac{n(n-1)}{1.2^2} + \frac{n(n-1)(n-2)}{1.2.3} - \&c.$$ 80

2930. Given
$$x+y+z-\xi-\eta-\zeta = i+j+k = i'+j'+k',$$
$$xy+xz+yz-\xi\eta-\xi\zeta-\eta\zeta = i(y+z)+j(z+x)+k(x+y)$$
$$= i'(\eta+\zeta)+j'(\zeta+\xi)+k'(\xi+\eta),$$
$$xyz-\xi\eta\zeta = iyz+jzx-kxy = i'\eta\zeta+j'\zeta\xi+k'\xi\eta;$$
prove that
$$\frac{(x-y)^2(x-z)^2(y-z)^2}{(\xi-\eta)^2(\xi-\zeta)^2(\eta-\zeta)^2} = \frac{i'j'k'}{ijk},$$
and that, in like manner, if there be any number of quantities $x, y, z, t \ldots$, and an equal number $\xi, \eta, \zeta, \tau \ldots$, connected together in a manner similar to the above, then
$$\frac{(x-y)^2(x-z)^2(x-t)^2(y-z)^2(y-t)^2(z-t)^2}{(\xi-\eta)^2(\xi-\zeta)^2(\xi-\tau)^2(\eta-\zeta)^2(\eta-\tau)^2(\zeta-\tau)^2} = \frac{i'j'k'l'\ldots}{ijkl\ldots}$$ 48

2932. Given the inscribed and circumscribed circles of a triangle, the envelope of the polar circle is a bicircular quartic. 52

2938. Let X, Y, Z be the intersections of the three pairs of lines through four points A, B, C, D. Then a line in their plane contains a pair of points harmonic conjugates to the conics through ABCD. Show that the locus of such points on the tangents to a curve of the kth class is a curve of the $(3k)$th order, having multiple points of the order k at each of the seven points A, B, C, D, X, Y, Z, the tangents at each of these points being easily found. If the first curve touch a line through two of the points A, B, C, D, the second curve contains this line. If the first curve is a conic through the same four points, the second curve contains the conic. 112

2942. Let p, q be the foci, and P, Q the asymptotes of a conic; θ the angle it subtends at a point a, and [A] the chord it cuts off from a line A. Then

1. If a line B is drawn through the point a meeting the conic in l, m,
$$al \cdot am \cdot \sin BP \sin BQ = \frac{ap^2 \cdot aq^2 \cdot \sin^2\theta}{pq^2}.$$

2. If from a point b on the line A tangents L, M are drawn to the conic,
$$\sin AL \sin AM \cdot bp \cdot bq = \frac{\sin^2 AP \sin^2 AQ \cdot [A]^2}{\sin^2 PQ}.$$

(Here al means the distance between the points a, l, and BP means the angle between the lines B, P.)

3. Find analogous propositions for a curve of any order on a plane or on a sphere. 99

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2946. To find three whole numbers in arithmetical progression whose common difference shall be a cube, the sum of any two diminished by the third a square, and the sum of the roots of the required squares a square. 91

2950. If j be any positive integer, then

$$\frac{j+1}{1.j}+\frac{j+1}{2(j-1)}+\frac{j+1}{3(j-2)}+\ldots\ldots+\frac{j+1}{j.1}=2\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{j}\right).$$ 56

2951. A given closed curve, whose equation is $\rho = f(\phi)$, moves without rotation so as always to subtend 90° at a fixed point A. In any position let ABCD be a circumscribed rectangle; show that the envelope of the sides BC or CD will be given by the equation $\rho = f(\phi+\frac{1}{2}\pi)+f(\phi-\frac{1}{2}\pi)$ 77

2952. Find the locus of points from which if a heavy inelastic globule be dropped on a smooth inclined plane, its velocity when passing a given horizontal line drawn on the plane shall be constant. 87

2955. 1. If T, T′ are tangents drawn from any point to the involute of a circle (radius = 1), and Σ the arc they intercept on the curve, then $T^2-T'^2 = 2(T+T'-\Sigma)$.

2. Let AOB, A′OB′ be two concentric quadrants, whose sides coincide. If the points A, B′ be joined by the arc AB′ of an involute of a circle, which arc touches the circular arcs at those points, the arc AB′ is an arithmetical mean between the circular arcs AB, A′B′. 74

2960. The envelope of a series of surfaces of order n, such that two of them can be drawn through an arbitrary point, is a surface of order $2n$, whose equation may be written in the form $\sqrt{(\alpha X)}+\sqrt{(\beta Y)}+\sqrt{(\gamma Z)}=0$, where $X=0$, $Y+0$, $Z+0$ are equations of any three surfaces of the series.

The envelope of a net of surfaces of order n, such that two of them can be drawn through *two* arbitrary points, is a surface of order $2n$, whose equation referred to any four surfaces of the net is of the same form as the equation of a quadric referred to four tangent planes. 95

2963. At the points where a transversal meets a quartic curve four tangents are drawn, which in general meet the curve again in eight points lying on a conic. Now suppose the quartic curve to be reduced to two conics, and the eight-point conic to become two lines: prove that the envelope of the transversal is a curve of the second class, $U=0$, and show that the discriminant of U is $\Delta^2\Delta_1^2(\Theta\Theta_1-\Delta\Delta_1)^2$ expressed in terms of the invariants of the two conics. 93

2964. The average focal radius-vector of (1) an ellipse, (2) a prolate spheroid, is half the major axis, the distribution being proportional in the former to the arc, in the latter to the surface..... 102

2972. OB and OC are any two semi-diameters of an ellipse conjugate to each other; find (1) the locus, and (2) the area, of the curve which is the intersection of normals at B and C. 106

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2977. Let x be any positive integer, and $1.3.5 \ldots (2x-1) = P_x$. Required to prove that

$$xP_{x-1} + \frac{x(x-1)}{2} 2P_{x-2} + \frac{x(x-1)(x-2)}{3} 2^2P_{x-3} + \ldots + \frac{x(x-1)\ldots 1}{x} 2^{x-1}$$
$$= P_x \left(1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2x-1}\right). \quad \ldots\ldots\ldots\ldots\ldots 83$$

2981. If the sum of the squares of two consecutive integers be equal to the square of another integer, find their general values, and show how to find any number of particular solutions. 104

2993. Required the average area of all the right-angled triangles whose hypotenuses are equal to $2a$..... 101

2995. From the top of a tower of height h, particles are projected in all directions in space, with a velocity due to a fall through a height h; show that the mean value of the range is given by the expression $2h\int_0^1 (1-x^4)^{\frac{1}{2}}\,dx$. 96

2996. If from the three angles of a triangle, which is circumscribed by a conic, lines be drawn parallel to those joining the middle points of the sides opposite those angles respectively with the centre of the conic, the three lines so drawn will cointersect. ... 85

2998. If two tangents to a cycloid include a constant angle, show that their sum has a constant ratio to the included arc of the curve... 107

3003. Prove that a uniform circular plate of matter attracting according to the inverse fifth power of the distance will serve to maintain the motion of a particle in the orbit of any circle cutting the circumference of the plate orthogonally, or to speak more strictly, in the portion of such orbit exterior to the plate. 92

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

2802. (Proposed by Professor SYLVESTER.)—Prove that, out of a set of numbers consisting of the first n coefficients in $(1+x)^{2n-1}$, or of the first n coefficients and the half of the middle coefficient in $(1+x)^{2n}$, terms may be selected, taken some positively and others negatively, so as to produce the sum 2^{n-1}. For example, if $n=6$, out of the set 1, 11, 55, 165, 330, 462 we may extract $-462+330+165-1=32$,

and out of the set 1, 12, 66, 220, 495, 792, $\frac{1}{2}(924)$ we may extract

$$495-\tfrac{1}{2}(924)-1=32.$$

Solution by the Rev. J. WOLSTENHOLME, M.A.

If $a_0, a_1, a_2, \ldots$ be the coefficients of the expansion of $(1+x)^{2n-1}$, and $i=\sqrt{(-1)}$ as usual,

$$(1+i)^{2n-1}=a_0-a_2+a_4-\ldots+i(a_1-a_3+\ldots),$$

$$\therefore (1+i)^{2n-1}+(1-i)^{2n-1}=2(a_0-a_2+a_4-\ldots)=2^{\frac{2n-1}{2}}\left(2\cos(2n-1)\frac{\pi}{4}\right),$$

$$\therefore a_0-a_2+a_4-a_6+\ldots=2^{n-1}\left(\cos\frac{n\pi}{2}+\sin\frac{n\pi}{2}\right)=\pm 2^{n-1};$$

and any of the coefficients a_{n-1+r} is equal to the corresponding coefficient a_{n-r}; so that from the first n coefficients of the expansion of $(1+x)^{2n-1}$ terms may be selected, taken some positively and others negatively, so as to produce 2^{n-1}. So if $(1+x)^n=a_0+a_1x+\ldots+a_rx^r+\ldots$ &c.,

$$(1+i)^{2n} \pm (1-i)^{2n} = 2(a_0 - a_2 + a_4 - \ldots) \text{ or } 2i(a_1 - a_3 + a_5 - \ldots)$$
$$= 2^n\left(2\cos\frac{n\pi}{2}\right) \text{ or } -2^n\left(2i\sin\frac{n\pi}{2}\right),$$

$$\therefore\ 2^n\cos\frac{n\pi}{2} = a_0 - a_2 + a_4 - \ldots,\quad 2^n\sin\frac{n\pi}{2} = -(a_1 - a_3 + a_5 - \ldots),$$

$$\therefore\ 2^n\left(\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right) = a_0 + a_1 - a_2 - a_3 + a_4 + a_5 - a_6 - \ldots.$$

In the right-hand side, substituting a_0 for a_{2n}, a_1 for a_{2n-1}, a_2 for a_{2n-2} &c., it is seen that some destroy each other, and some repeat; but a_n, the coefficient of the middle term, appears only once, either positively or negatively. Hence $2^{n-1}\left(\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right)$, or $\pm 2^{n-1}$ can be formed by selecting terms from the first n coefficients and half the middle coefficient of $(1+x)^{2n}$, to be taken either positively or negatively.

2884. (Proposed by MATTHEW COLLINS, B.A.)—Subtract each of the seven Dominical Letters A, B, C, D, E, F, G from the sum of the other six of them, and then find the coefficient of the term ABCDEFG in the continued product of the seven remainders thus obtained.

Solution by the REV. J. L. KITCHIN, M.A.

Take, for generality, n letters, and put S for their sum; then the factors similar to those in the question are

$$S-2A,\quad S-2B,\quad S-2C,\quad \ldots,\quad S-2L;$$

and the product is easily seen to be

$$S^n - 2(A+B+C+\ldots+L)S^{n-1} + 2^2(A.B+A.C+B.C+\ldots)S^{n-2} - \ldots$$
$$-2^3(A.B.C+A.C.D+B.C.D+\ldots+)S^{n-3} + \ldots + (-2)^n A.B.C\ldots L.$$

Now the term in S^n containing ABC...L has for coefficient, by the multinomial theorem, $\frac{\lfloor n}{\lfloor 1\ \lfloor 1} = \lfloor n$.

Similarly, the coefficient of BC...L in S^{n-1} $= \lfloor n-1$,
and the coefficient of AC...L „ $= \lfloor n-1$,
&c. &c. $=$ &c.

And there are n such sets, each multiplied by A, B, &c., in order; in all $= \lfloor n$.

In S^{n-2}, the coefficient of CD...L $= \lfloor n-2$,
&c. &c. $=$ &c.,

and there are $\frac{n(n-1)}{1.2}$ such sets corresponding in order with the terms in $AB+AC+\ldots$

therefore coefficient of ABC... L in this set $= \frac{\lfloor n}{1.2}$, and so on;

therefore coefficient of ABC... L

$$= \underline{\lfloor n} \left\{1-2+\frac{2^2}{1.2}-\frac{2^3}{1.2.3}+\cdots+\frac{(-2)^n}{\underline{\lfloor n}}\right\} = \underline{\lfloor n} \cdot \epsilon_n^{-2},$$

adopting the notation lately introduced in this Journal.

If $n=7$, the coefficient is easily found to be 656.

2841. (Proposed by A. MARTIN.)—A given cylindrical vessel, filled with water, is placed with its base upon a horizontal plane. It is required to find the angle of inclination to which the plane must be raised before the vessel will fall, the water being at liberty to overflow the top. The base is supposed to be fixed so as to prevent it from sliding, but not from tilting, when the plane is inclined.

Solution by J. J. WALKER, M.A.

Let h be the height of the cylinder, a the radius of base, θ the inclination at which tilting takes place, x, y the distances of the centre of gravity of the fluid remaining in the cylinder from axis and base. Then the condition

$$x + y\tan\theta = a \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

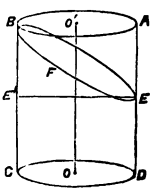
evidently expresses that the vertical through the centre of gravity meets the circumference of the base. It will be found that

$$x = \frac{a^2\tan\theta}{4(h-a\tan\theta)}, \quad y = \frac{4h^2-8ah\tan\theta+5a^2\tan^2\theta}{8(h-a\tan\theta)},$$

whence (1) becomes

$$5a^2\tan^3\theta - 8ah\tan^2\theta + (10a^2+4h^2)\tan\theta - 8ah = 0.$$

If $h=2a$, this equation will have a single real root lying between $\tan\theta = 1$ and $\tan\theta = 0{\cdot}9$.



To prove the above results, let ABCD be a cylinder, and let any ungula be cut off from it by a plane BFE making an angle θ with the base. The volume of the cylindrical wedge BE'E (its base being parallel to that of the cylinder) is equal to $\pi a^3\tan\theta$. Its moment with respect to a plane through the axis OO', perpendicular to EE', is equal to $\frac{1}{4}\pi a^4\tan\theta$. Its moment with respect to its base is equal to $\frac{5}{4}\pi a^4\tan^2\theta$. From these the distances of the centre of gravity of the ungula from the axis and base of the cylinder are readily found to have the values given above.

2829. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—Through the fourth point of intersection of two conics circumscribing the triangle of reference (whose equations are $l\beta\gamma+m\gamma\alpha+n\alpha\beta=0$, $l'\beta\gamma+m'\gamma\alpha+n'\alpha\beta=0$ is drawn a straight line meeting the curves again in P, P': prove that the tangents at P, P' intersect on the tri-cusped quartic

$$\left\{\beta\gamma\left(\frac{1}{mn'}-\frac{1}{m'n}\right)\right\}^{\frac{1}{2}}+\left\{\gamma\alpha\left(\frac{1}{nl'}-\frac{1}{n'l}\right)\right\}^{\frac{1}{2}}+\left\{\alpha\beta\left(\frac{1}{lm'}-\frac{1}{l'm}\right)\right\}^{\frac{1}{2}}=0.$$

I. *Solution by* STEPHEN WATSON.

Put $\quad \mathrm{L}=mn'-m'n, \quad \mathrm{M}=nl'-n'l, \quad \mathrm{N}=lm'-l'm,$

$$\mathrm{L}'=\frac{1}{mn'}-\frac{1}{m'n}, \quad \mathrm{M}'=\frac{1}{nl'}-\frac{1}{n'l}, \quad \mathrm{N}'=\frac{1}{lm'}-\frac{1}{l'm};$$

then the given conics may be written

$$na\,(\mathrm{M}\beta-\mathrm{N}\gamma)=l\gamma\,(\mathrm{L}\alpha-\mathrm{M}\beta) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$n'\alpha\,(\mathrm{M}\beta-\mathrm{N}\gamma)=l'\gamma\,(\mathrm{L}\alpha-\mathrm{M}\beta) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and a line through their fourth point is

$$\mathrm{M}\beta-\mathrm{N}\gamma=k\,(\mathrm{L}\alpha-\mathrm{N}\beta);$$

hence the point P is determined by

$$\frac{k\alpha}{l}=-\frac{(k+1)\beta}{m}=\frac{\gamma}{n},$$

and the tangent at P is $\dfrac{k^2\alpha}{l}+\dfrac{(k+1)^2\beta}{m}+\dfrac{\gamma}{n}=0$ $\ldots\ldots\ldots\ldots\ldots\ldots (3).$

In like manner the tangent at P' is

$$\frac{k^2\alpha}{l'}+\frac{(k+1)^2\beta}{m'}+\frac{\gamma}{n'}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Eliminate first k^2 and then $(k+1)^2$ from (3) and (4), the results give

$$k+1=\pm\left(\frac{\mathrm{M}'\gamma}{\mathrm{N}'\beta}\right)^{\frac{1}{2}} \quad \text{and} \quad k=\pm\left(\frac{\mathrm{L}'\gamma}{\mathrm{N}'\alpha}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (5),$$

and the final result may be written

$$(\mathrm{L}'\beta\gamma)^{\frac{1}{2}}+(\mathrm{M}'\gamma\alpha)^{\frac{1}{2}}+(\mathrm{N}'\alpha\beta)^{\frac{1}{2}}=0,$$

as it is plain the signs must be so taken in (5) as to render this final result uniform.

II. *Solution by* R. TUCKER, M.A.

Through the fourth point of intersection O $(\alpha'\beta'\gamma')$ draw the straight line OPP' cutting the conics in P$(\alpha_1\beta_1\gamma_1)$ and P'$(\alpha_2\beta_2\gamma_2)$; then we have the equations

$$\frac{l\alpha}{\alpha'\alpha_1}+\frac{m\beta}{\beta'\beta_1}+\frac{n\gamma}{\gamma'\gamma_1}=0, \quad \frac{l'\alpha}{\alpha'\alpha_2}+\frac{m'\beta}{\beta'\beta_2}+\frac{n'\gamma}{\gamma'\gamma_2}=0;$$

and from the identity of these lines we get

$$\frac{l\alpha_2}{l'\alpha_1}=\frac{m\beta_2}{m'\beta_1}=\frac{n\gamma_2}{n'\gamma_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The tangents at P and P′ are given by the equations

$$\frac{la}{a_1^2}+\frac{m\beta}{\beta_1^2}+\frac{n\gamma}{\gamma_1^2}=0\ldots\ldots(2),\quad \frac{l'a}{a_2^2}+\frac{m'\beta}{\beta_2^2}+\frac{n'\gamma}{\gamma_2^2}=0\ldots\ldots(3);$$

this latter, by virtue of (1), becomes $\frac{l^2a}{l'a_1^2}+\frac{m^2\beta}{m'\beta_1^2}+\frac{n^2\gamma}{n'\gamma_1^2}=0\ldots\ldots(4)$;

and we have further $\frac{l}{a_1}+\frac{m}{\beta_1}+\frac{n}{\gamma_1}=0\ldots\ldots(5)$.

Eliminating a_1, β_1, γ_1 between (2), (4), and (5), we obtain the required equation, since $\frac{l'}{l}\cdot\frac{a_1^2}{a}(m'n-mn')=\frac{m'}{m}\cdot\frac{\beta_1^2}{\beta}(ln'-nl')=\frac{n'}{n}\cdot\frac{\gamma_1^2}{\gamma}(l'm-lm')$.

2608. (Proposed by S. ROBERTS, M.A.)—Show that if (a_1, a_2) are points of contact of tangents from a point A on a cubic curve, and (b_1, b_2) similar points of contact of tangents from B, another point on the same curve; and if (a_1, b_1) and (a_2, b_2) respectively are collinear with the point of contact of a tangent from the point where the line AB meets the curve again, then $A B a_1 a_2 b_1 b_2$ lie in the same conic.

I. *Solution by* T. COTTERILL, M.A.

Generally, if (a_1, b_1), (a_2, b_2) are points on a cubic such that the lines through (a_1, b_1) and (a_2, b_2) meet in c on the curve, then the opposite of the four points (a_1, b_1), (a_2, b_2) is C, the tangential of c; or in other words, a conic can be drawn through (a_1, b_1), (a_2, b_2) and the two remaining points of intersection of any line through C with the cubic. But in the question, A and B are two such points, because C, A, B are the tangentials of three points c, a_1, b_1 or c, a_2, b_2, either triad being collinear points on the cubic.

II. *Solution by* W. S. MCCAY, B.A.

It is shown in Salmon's *Higher Plane Curves*, that the three intersections

Aa_1 and Bb_2 ....(1), Aa_2 and Bb_1 ....(2), a_1b_1 and a_2b_2 ....(3)

are in one straight line. But this is the condition that the six vertices of the hexagon $Aa_1b_1Bb_2a_2$ should lie on a conic—the vertices being taken in this order, for then Pascal's theorem holds for the intersections of $\begin{vmatrix} 12 \,\&\, 45 \\ 23 \,\&\, 56 \\ 34 \,\&\, 61 \end{vmatrix}$.

2793. (Proposed by W. K. CLIFFORD, B.A.)—C is the single focus of a semicubical parabola, and from any point O three tangents are drawn to the curve; if CD, CE, CF be perpendicular to them, show that DE and CF are equally inclined to the direction of the infinite branches.

Solution by J. J. WALKER, M.A.

Let $y^2=4ax$ be the parabola which has the given semi-cubical parabola as its evolute; then, if (ξ, η) be the point D, *i. e.* the point in which a perpendicular from focus meets the normal to the parabola at (x', y'), $\xi = x'+a$, $\eta = \frac{1}{2}y'$. Similarly for E, $\xi' = x''+a$, $\eta' = \frac{1}{2}y''$; so that the tangent of the angle which the line DE makes with the specified direction is

$$\frac{\xi-\xi'}{\eta-\eta'} = \frac{2(x'-x'')}{y'-y''} = \frac{y'+y''}{2a}.$$

Let O be (X, Y), then if (x, y) be one of the three points normals at which meet in O,

$$xy+(2a-X)y-2aY=0;$$

and eliminating x between this and $y^2=4ax$, we have a cubic in y wanting the second term. Hence, if (x''', y''') be the point at which the third normal through O meets the parabola $\frac{y'+y''}{2a} = -\frac{y'''}{2a}$, and $\frac{y'''}{2a}$ is the tangent of inclination to the specified direction of the tangent to the parabola at (x''', y'''), that is, of the line CF.

2801. (Proposed by M. JENKINS, M.A.)—Show that the number of pairs of numbers which have a given number G for their G. C. M., and another given number L (of course a multiple of G) for their L. C. M., is 2^{n-1}, where n is the number of prime bases the product of whose powers is equal to $\left(\frac{L}{G}\right)$.

I. *Solution by the* Rev. J. WOLSTENHOLME, M.A.

Let xG, yG be a pair of numbers whose L. C. M. is L and G. C. M. is G; then x, y must be prime to each other, and $xy = \frac{L}{G}$. If, then, this latter number be $a^p b^q c^r \ldots$ where a, b, $c \ldots$ are prime, the values of x satisfying these conditions are the different terms of the expansion of $(1+a^p)(1+b^q)(1+c^r)\ldots$, 2^n in number. But since the same pair of numbers are given by x and by $\frac{L}{Gx}$, the number of different pairs of numbers is 2^{n-1}.

II. *Solution by the* PROPOSER.

If Ga and Gb be any pair, a must be prime to b, and $L = Gab$, or $ab = \left(\frac{L}{G}\right)$; and the number of different pairs a, b is therefore the number of ways of separating $\left(\frac{G}{L}\right)$ into conjugate divisors which are prime to each other. This number is well known to be 2^{n-1}.

2542. (Proposed by S. ROBERTS, M.A.)—A given conic is continually touched by a similar and similarly placed conic; find the locus of a point rigidly connected with the moving conic.

Solution by the Rev. J. WOLSTENHOLME, M.A.

Let $ax^2+by^2=1$ be the fixed conic, $a(x-h)^2+b(y-k)^2=m$ the moving one, $X=h+\alpha$, $Y=k+\beta$ the coordinates of a point rigidly connected with the moving conic. For tangency,

$$4(ah^2+bk^2)=(1+ah^2+bk^2-m)^2; \quad \text{or} \quad ah^2+bk^2=1+m\pm 2\sqrt{m};$$

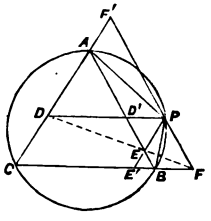
and the locus of the point is the two conics

$$a(X-\alpha)^2+b(Y-\beta)^2=(1\pm\sqrt{m})^2,$$

clearly as the contact is internal or external.

2849. (Proposed by R. TUCKER, M.A.)—An ellipse is described about a triangle having the *centroid* for its centre; show that straight lines from any point on it parallel to the sides cut the sides in two sets of three points which are collinear.

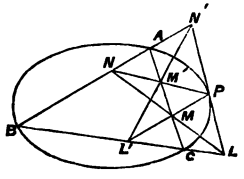
I. *Solution by* J. J. WALKER, M.A.



The triangle being concentroidal with the ellipse, the tangents at vertices will be parallel to the opposite sides; hence, if the ellipse be projected orthogonally into a circle, the triangle will be projected into an inscribed equiangular triangle.

To prove the property stated in the projected figure: let ABC be an equiangular triangle, and P any point on the circumscribing circle; and let parallels to BC, AC, AB, drawn from P, meet CA, AB, BC in D, E, F, and AB, BC, CA in D′, E′, F′. Join DE, EF, PA, PB. Then the points A, P, E, D lie on a circle; therefore $\angle PED + \angle PAD = 2$ right angles; whence $\angle PED = \angle PBC$. Again, the points P, F, B, E lie on a circle, whence $\angle PBC = \angle FEE'$; therefore $\angle FEE' = \angle PED$, and DEF is a straight line. Similarly it may be shown that D′, E′, F′ are collinear.

II. *Solution by* STEPHEN WATSON.



Let ABC be the triangle; take BC ($=a$) and BA ($=c$) for axes, and denote any point P by (x, y). Then, if through F lines LPN′, PML′, PM′N be drawn parallel respectively to AC, AB, BC, cutting the sides as in the diagram, the coordinates of L, M, N are

$$\left(\frac{cx+ay}{c}, 0\right), \left(x, \frac{c(a-x)}{a}\right), \text{ and } (0, y);$$

and the condition that those three points lie in a straight line is

$$c^2x^2 + a^2y^2 + acxy - ac(cx + ay) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

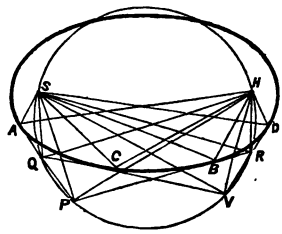
Similarly, that L′, M′, N′ lie in a straight line, gives the same result; moreover, (1) is an ellipse having its centre at the point $(\frac{1}{3}a, \frac{1}{3}c)$, or the centroid of the triangle ABC; hence the property stated in the question is established.

2840. (Proposed by F. D. THOMSON, M.A.)—A circle is described through the foci of a conic, and from a point P on it tangents PA, PB are drawn to the curve, meeting the circle again in Q and R. Show that the other tangents to the conic from Q and R meet on the circle.

Solution by the PROPOSER.

1. We have to show that, if V be the point where the tangents meet, the angle QSV is equal to the angle QHV.

Now, with the notation of the figure, remembering that a pair of tangents subtend equal angles at either focus,



$$\begin{aligned} QSV &= QSC + CSV = QSA + DSV \\ &= PSA - PSQ + DSR + RSV \\ &= PSB - PHQ + BSR + RSV \\ &= PSR - PHQ + RSV \\ &= PHR - PHQ + RSV. \end{aligned}$$

$$\begin{aligned} \text{Also } QHV &= QHC + CHV = QHA + DHV = PHA - PHQ + DHR + RHV \\ &= PHB - PHQ + BHR + RHV = PHR - PHQ + RHV; \end{aligned}$$

therefore $QSV - QHV = RSV - RHV,$

therefore $QSV - RSV = QHV - RHV.$

But $QSV + RSV = QSR = QHR = QHV + RHV;$

therefore $QSV = QHV.$

2. The above is a particular case of the general theorem, that if two conics are such that a quadrilateral may be inscribed in one and circumscribed about the other, then an infinite number of such quadrilaterals may be found.

In this case, the conic touches the quadrilateral formed by the tangents through the foci, and the circle passes through the foci and the circular points, that is, circumscribes the same quadrilateral.

The condition that two conics may be so related, may be obtained by forming the invariants of the two conics referred to the triangle formed by the diagonals. Their equations are

$$(x + y + z)(y + z - x) + \kappa(z + x - y)(x + y - z) = 0 \quad \ldots\ldots\ldots (1),$$

and

$$ax^2 + by^2 + cz^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

with the condition that (2) touches the sides of the quadrilateral, *i. e.*

$$bc + ca + ab = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

(1) may be written $\left(1,\ -1,\ -1,\ \frac{\kappa+1}{\kappa-1},\ 0,\ 0\right)(x, y, z)^2 = 0,$

of which the tangential equation is

$$\left(1-\left(\frac{\kappa+1}{\kappa-1}\right)^2,\ -1,\ -1,\ -\frac{\kappa+1}{\kappa-1},\ 0,\ 0\right)(\lambda, \mu, \nu)^2 = 0.$$

(2) has for its tangential equation

$$(bc,\ ca,\ ab,\ 0,\ 0,\ 0)(\lambda\mu\nu)^2 = 0.$$

Therefore, with the notation of Salmon's *Conics*,

$$\Delta = 1-\left(\frac{\kappa+1}{\kappa-1}\right)^2, \quad \Delta' = abc,$$

$$\Theta = a\left[1-\left(\frac{\kappa+1}{\kappa-1}\right)^2\right]-b-c = a\Delta-b-c, \quad \Theta' = bc-ca-ab.$$

But, from (3), $a = -\frac{bc}{b+c}$; hence we have

$$\Delta' = -\frac{b^2c^2}{b+c}, \quad \Theta = -\frac{bc}{b+c}\Delta-(b+c), \quad \Theta' = 2bc;$$

therefore

$$\Theta\Theta'-2\Delta'\Delta-\frac{\Theta'^3}{4\Delta'} = 0,$$

a homogeneous relation between the invariants, and therefore the condition required.

3. If the points P and Q coincide, PA is a common tangent, and the points V and R will coincide, and therefore VD will be another common tangent. Hence the line joining the points of contact of common tangents touches the conic, and therefore by symmetry is the tangent at the end of the minor axis. This gives a solution of Mr. WOLSTENHOLME'S Question 2744.

II. *Solution by* R. TUCKER, M.A.

From any point P on the focal circle draw a pair of tangents to the conic, and from one of the points of section with the circle draw a third tangent cutting the circle in Q, and join Q with the remaining point of section, and produce the line. If now $Y_1, Y_2, Y_3, Y_4, Z_1, Z_2, Z_3, Z_4$ be the feet of perpendiculars from S and H on the three tangents and the line through Q, we have

$$SY_1 . SY_3 = \pm SY_2 . SY_4, \quad HZ_1 . HZ_3 = \pm HZ_2 . HZ_4.$$

Multiplying these equations, and remembering that $SY . HZ = b^2$, we have

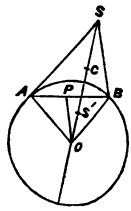
$$b^4 = b^2 . SY_4 . HZ_4, \quad i.e., \quad SY_4 . HZ_4 = b^2;$$

hence, as may readily be shown by the aid of this result, the line through Q is a tangent to the conic. This proves the property of the question.

NOTE.—$SY_1 . SY_3 = \pm SY_2 . SY_4$ by the equation $\alpha\gamma = \pm\beta\delta$ for a circle. Also, if product of perpendiculars from foci on the line $y = mx + c$ be $= b^2$, we have $c^2 - m^2a^2e^2 = b^2(1+m^2)$, therefore $c^2 = b^2 + m^2a^2$, the condition for tangency.

2684. (Proposed by MORGAN JENKINS, M.A.)—From a point S two straight lines are drawn meeting a circle in A and B, so that the rectangle SA . SB is constant; show that the envelope of the secant AB is a central conic of which S and its conjugate point are foci; and that the conic is an ellipse or hyperbola according as the given rectangle is greater or less than the rectangle contained by the intercepts made by the circle on any straight line through S.

I. *Solution by the* PROPOSER.



Let O be the centre of the circle; then, taking OS for initial line, and putting θ, θ' for the angles AOS, BOS, p for the perpendicular from O on AB, ϕ for the angle between AB and OS, $OA = c$, $OS = k$, and l^2 = the given rectangle, we have

$$p = c\cos\tfrac{1}{2}(\theta-\theta'), \quad \phi = \tfrac{1}{2}\pi + \tfrac{1}{2}(\theta+\theta'),$$
$$(c^2+k^2-2ck\cos\theta)(c^2+k^2-2ck\cos\theta') = l^4.$$

Eliminating θ and θ', we shall obtain

$$\left(p - \frac{c^2+k^2}{2k}\sin\phi\right)^2 + \left(\frac{c^2-k^2}{2k}\right)^2\cos^2\phi = \left(\frac{l^2}{2k}\right)^2.$$

This is the equation of a central conic referred to the major axis as initial line and a point on it as origin. The distances of the centre of the conic from the origin and from the focus are $\frac{c^2+k^2}{2k}$ and $\frac{c^2-k^2}{2k}$ respectively; whence the distances of the foci from the origin O are k and $\frac{c^2}{k}$ respectively; hence S and its conjugate point are the foci. Also $\frac{l^2}{2k}$ is the semi-major axis; hence the conic is an ellipse or hyperbola according as $c^2 \lessgtr k^2$ is $<$ or $> l^2$.

II. *Solution by* J. J. WALKER, M.A.

Let O be the centre of the given circle, P any point in AB; also let $OA = a$, $OS = b$, $\angle AOS = \phi$, $\angle BOS = \phi'$, $\angle POS = \theta$, $AS : SB = c^2$. For shortness, let $\cos\frac{1}{2}(\phi+\phi') = u$, $\cos\frac{1}{2}(\phi-\phi') = v$, $a^2+b^2 = h^2$, $2ab = h'^2$, and $(a^2-b^2)^2 - c^4 = k^4$.

Then it is easily found that the polar equation to the secant AB may be thrown into the form

$$r^2\sin^2\theta(h'^4u^2 - 2h^2h'^2uv + h^4 - c^4) - h^4(r\cos\theta\, v - au)^2 = 0 \quad \ldots\ldots (1),$$

while, by the terms of the question, the parameters u, v are connected by the equation

$$h'^4(u^2+v^2) - 2h^2h'^2uv + k^4 = 0 \quad \ldots\ldots\ldots\ldots\ldots (2).$$

Differentiating (1) and (2) with respect to u and v, and eliminating du, dv between the results, we obtain

$$a(ah^2 - rh'^2\cos\theta)u^2 + (r^2-a^2)h'^2uv + r(ah'^2\cos\theta - rh^2)v^2 = 0 \quad \ldots\ldots (3).$$

Again, subtracting (1) multiplied by k^4 from (2) multiplied by $r^2\sin^2\theta(h^4-c^4)$,

$$(r^2h'^2\sin^2\theta + a^2k^4)u^2 - 2r(rh^2h'^2\sin^2\theta + ak^4\cos\theta)uv + r^2(h'^4\sin^2\theta + k^4)v^2 = 0$$
$$\ldots\ldots (4).$$

Finally, eliminating u, v between (3) and (4), the result may be reduced to the product of the factors

$$r^2 \sin^2\theta \left\{ [(a^2+r^2)b - (a^2+b^2) r\cos\theta]^2 + (a^2-b^2)^2 r^2 \sin^2\theta \right\},$$

and $$k^8 + 4b\left\{(a^2+r^2)b - (a^2+b^2) r\cos\theta\right\} k^4 - 4(a^2-b^2)^2 b^2 r^2 \sin^2\theta.$$

The first factor, being independent of k, is irrelevant. The second, equated to zero, is the required envelope; which, if $2bx = 2br\cos\theta - (a^2+b^2)$, $y = r\sin\theta$, becomes $4b^2(k^4x^2 - a^4y^2) = a^4k^4$, a hyperbola if $a^2 \curvearrowright b^2 > k^2$, an ellipse if $a^2 \curvearrowright b^2 < k^2$. The foci are S and its reciprocal point with respect to the circle. If $a^2 \curvearrowright b^2 = k^2$, the secant AB passes either through S or S′. If S is on the circle, S and S′ coincide, and the ellipse becomes a circle, the solution of Quest 2100. (*Reprint*, Vol. X., p. 97.)

2747. (Proposed by W. S. McCay, B.A.)—Prove that the locus of the foot of the bifocal perpendicular on the tangent plane to a quadric is a concentric sphere whose radius is the primary axis.

Solution by the Proposer.

For the foot of this perpendicular is a point through which can be drawn three rectangular tangent planes to three confocals. Since the two tangent planes through a line to the two confocals that it touches are at right angles, the locus is a sphere. (See Salmon's *Geometry of three Dimensions*, Arts. 168, 172.)

2894. (Proposed by the Rev. J. Wolstenholme, M.A.)—An ellipse and hyperbola are confocal, and the asymptotes of the latter lie on the equi-conjugates of the former; prove that any ellipse which passes through the ends of the axes of the given ellipse will cut the hyperbola orthogonally.

Solution by W. H. H. Hudson, M.A.

If $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the hyperbola, the equation of an ellipse confocal and with its equi-conjugates along the asymptotes may be found to be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2+b^2}{a^2-b^2}.$$

Any ellipse through the intersection of this with $xy = 0$ may be taken to be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + 2\lambda xy = \frac{a^2+b^2}{a^2-b^2}.$$

If this intersect the hyperbola in (x, y), the tangents to the two curves at the point of section are

$$\frac{Xx}{a^2} - \frac{Yy}{b^2} = 1, \quad X\left(\frac{x}{a^2} + \lambda y\right) + Y\left(\frac{y}{b^2} + \lambda y\right) = \frac{a^2 + b^2}{a^2 - b^2}.$$

Now, since (x, y) is in both curves,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + 2\lambda xy = \frac{a^2 + b^2}{a^2 - b^2}\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right);$$

and this will be found to be the same as the condition that the two tangents are at right angles, viz., $\frac{x}{a^2}\left(\frac{x}{a^2} + \lambda y\right) - \frac{y}{b^2}\left(\frac{y}{b^2} + \lambda x\right) = 0.$

2782. (Proposed by M. W. CROFTON, F.R.S.)—1. A given point is known to be within a certain circle of given radius, but unknown position; find the chance that another given point is also within the circle.

2. Three given points are known to be within a certain circle, which is otherwise altogether unknown; determine the most probable position of its centre.

3. Two given points are known to be within a circle, and a third given point is known to be outside it; determine the most probable position of its centre.

Solution by the REV. J. WOLSTENHOLME, M.A.

1. If A, B be the two given points, $AB = 2c$, a the radius of the given circle, P its centre, then all the possible positions of P lie within the circle centre A and radius a; and in order that B may also lie within the circle, P must lie within the circle centre B and radius a; the probability is then obviously $\frac{\theta - \sin\theta\cos\theta}{\pi}$, where $\cos\theta = \frac{c}{a}$.

2. If x, y, z be the distances of the centre of the circle from three given points A, B, C, the radius of the circle must be greater than the greatest of the three x, y, z; the most probable position of the centre is then when this greatest of the three distances has its least value. If now x be the greatest of the three, through B′, C′, middle points of AC, AB, draw straight lines at right angles to these sides, meeting in O the centre of the circle circumscribing ABC. Then, in order that x may be $> y$ or z, the centre of the circle must lie in the angle vertically opposite B′OC′, and the least value will be when the centre is at O, if the angles B, C be acute. So if y, or z, be the greatest. Hence the most probable position of the centre, when the triangle is acute-angled, is O. If, however, one of the angles (A) be obtuse, the least value of x, when x is the greatest of the three, will be when the centre is at O, but the least value of y, (or z), when it is the greatest, will be when the centre is at A′, the middle point of the side opposite the obtuse angle, and since this least value is less than the former, A′ is the most probable position of the centre.

3. If A be the point without the circle, then if r be the radius, $x > r$, $y < r$,

$z < r$; hence the centre must lie in the angle vertically opposite B'OC', and the most probable position of the centre is that in which, if y be the greater of the two y, z, $x-y$ is the greatest possible, which, if OA' be drawn at right angles to BC and the angle C be obtuse, will be found when a hyperbola with foci A, B touches OA' with the branch to which B is an interior focus. If the angles be acute, it seems to me the most probable position must be at infinity on OA'.

2864. (Proposed by Professor SYLVESTER.)—A body moves in an ellipse with *uniform* velocity under the attraction of two central forces at the foci. Show that the forces to each focus are always equal, and vary inversely as the product of the focal distances.

Solution by J. J. WALKER, M.A.; JAMES DALE; *and others.*

Since the velocity is uniform, the resultant of the attractive forces must at every point be normal to the ellipse; therefore their tangential components must be equal and opposite; and the lines to the foci making equal angles with the tangent at the point, it follows that the forces to the two foci must be equal. Again, the normal force at each point must vary as the curvature, and therefore the force to either focus as the curvature directly, and the cosine of angle between normal and line to focus inversely; *i. e.*, in the sesquiplicate ratio of product of the focal distances inversely, and the subduplicate ratio of the same product directly, or inversely as that product.

2811. (Proposed by M. JENKINS, M.A.)—If

$$u = \sum_{s=0}^{s=p-2} \frac{\Pi(m+n)r}{\Pi\{mr+k(p-1)\}\ \Pi\{nr-k(p-1)\}} \equiv c \pmod{p},$$

where p is any odd prime number; m, n, and r any positive integers; and k takes all integral values between $\frac{nr}{p-1}$ and $-\frac{mr}{p-1}$, inclusive of those quantities, if integral; show that c has the same value for all values of r which have the same G. C. M. with $(p-1)$.

Solution by the PROPOSER.

In the series $v = \sum_{s=0}^{s=p-2} \theta^{(m+n)\,\mathrm{Ind}_\gamma(\gamma^s+1)-ms}$, where γ is any primitive root of p, and $\mathrm{Ind}_b a$ means the congruential logarithm of a to the base b, that is, $\mathrm{Ind}_b a = x$ if $b^x \equiv a \pmod{p}$, for θ put γ^r; and we obtain

$$\sum_{s=0}^{s=p-2} \frac{(\gamma^s+1)^{(m+n)r}}{\gamma^{mrs}} = \omega \text{ suppose.}$$

Expanding the numerator of ω and dividing by the denominator, we obtain a series of terms of the form $C\gamma^{\rho s}$; and $\sum_{s=0}^{s=p-2} \gamma^{\rho s} \equiv 0 \pmod{p}$ except when $\rho = 0$ or a multiple of $(p-1)$, in which case

$$\sum_{s=0}^{s=p-2} \gamma^{\rho s} \equiv -1 \pmod{p}.$$

Hence, collecting these terms, we find that $\omega \equiv -u \pmod{p}$, a result which is independent of the particular primitive root employed.

Since γ^r can be made $\equiv \beta^q$ or α^d, where d is the G. C. M. of r and $(p-1)$, q is any other integer having d for the G. C. M. of itself and $(p-1)$, and α, β are certain other primitive roots of p, the above-stated theorem follows.

I will add the following theorem:—If $(m+n)r$ be, and mr (or nr) be not, a multiple of $(p-1)$, then $u \equiv \pm 1 \bmod p$ according as mr is even or odd.

For the numerator in $\omega \equiv 1 \pmod{p}$ for all values of s except $\frac{p-1}{2}$; hence

$$\omega \equiv \left\{\sum_{s=0}^{s=p-2} \frac{1}{\gamma^{mrs}}\right\} - \frac{1}{\gamma^{mr\frac{p-1}{2}}} \equiv -\frac{1}{\gamma^{mr\frac{p-1}{2}}},$$

and $t \equiv \dfrac{1}{\gamma^{mr\frac{p-1}{2}}} \equiv \pm 1$ according as mr is even or odd.

2603. (Proposed by T. Cotterill, M.A.)—In a plane, (1) on every line, there are two points harmonic conjugates to the conics through four points ABCD. They are real, if the product of the four triangles ABC, BCD, CDA, DAB, and the perpendiculars from the points ABCD on the line, is positive. If a line does not cut a conic, why are the conjugates on the line real, to any set of four points on the conic? Give a rule for determining the nature of the points on a line, when the product of the triangles is (1) positive, (2) negative.

(2.) Through any point, there are two lines conjugate to the conics inscribed in four lines. They are real in those regions of the plane formed by the four lines in which the product of perpendiculars from the point has the same sign as that of points in the convex quadrilateral. Hence every point on or within a conic inscribed in the four lines lies in these regions.

Solution by the Rev. J. Wolstenholme, M.A.

The series of conics described through four given points cut any straight line in points in involution. The foci of this involution are, of course, the points of contact of the two conics through the four points touching the line; and these two points will therefore be conjugates with respect to any conic through the four points. The conditions of their reality are, (1) when the four points form a convex quadrangle, the straight line must be so situated as to have either all four points on the same side of it, or two on one side and two on the other; (2) when the four points form a reentrant

quadrangle, one of them must be on the opposite side of the straight line to the other three.

Of course, reciprocally, the tangents drawn from a point to a series of conics inscribed in a quadilateral form a pencil in involution; whose double rays, the tangents to the two conics which pass through the point, are conjugates with respect to any conic of the series. These will, therefore, be real if the point lie in the convex quadrilateral formed by the lines, or in any of the portions of space into which a point can pass from this by crossing two of the lines simultaneously.

I am not sure that I understand Mr. Cotterill's query. But, given a straight line and a conic which does not meet it in real points, it is manifest that whatever four points may be taken on the conic will either, in an ellipse and parabola, form a convex quadrangle and be all on one side of the straight line; or, in a hyperbola, whose branches must lie on opposite sides of the straight line, will form a convex quadrangle, and be all on one branch, or two on one branch and two on the other; or will form a reentrant quadrangle, and thus be three on one branch and one on the other. In all such cases, then, the conjugate points on the line, to all conics through the four points, will be real.

It may be noticed that in the first case the three pairs of straight lines through the four points are particular cases of the series, and cut the given straight lines in six points in involution whose foci are therefore readily found. Correspondingly, in the second case, the straight lines joining the given points to the ends of the diagonals of the quadrilateral form a pencil of six rays in involution and the double rays are known.

In considering this question, I discovered the property proposed for proof in Question 2875.

2735. (Proposed by R. TUCKER, M.A.)—The escribed radii of a triangle are in harmonical progression, and the mean escribed radius is a geometrical mean between the inscribed and circumscribed radii: show that the cosine of the mean angle of the triangle is $\frac{2}{3}$, and that the circumscribed radius is nine times the inscribed radius.

Solution by the PROPOSER.

Since the escribed radii are in harmonic progression, we have

$$a+c=2b, \quad \text{and} \quad 2s=3b;$$

also
$$\left(\frac{2\Delta}{b}\right)^2 = Rr = \frac{abc}{4\Delta}\cdot\frac{\Delta}{s} = \frac{abc}{6b} = \frac{ac}{6};$$

therefore
$$a^2\sin^2 C = \tfrac{1}{6}ac, \quad \text{or} \quad \sin A \sin C = \tfrac{1}{6}.$$

Now
$$1+\cos B = \frac{(a+c)^2-b^2}{2ac} = \frac{3b^2}{2ac} = 9\sin^2 B, \quad \text{hence} \quad \cos B = \tfrac{8}{9}.$$

Also
$$\frac{R}{r} = \frac{bs}{2\sin B\,.\,\Delta} = \frac{3b^2}{2ac\sin^2 B} = 9, \quad \text{therefore} \quad R = 9r.$$

We also get
$$\tan \tfrac{1}{2}a\,.\,\tan \tfrac{1}{2}C = \tfrac{1}{3}.$$

2582. (Proposed by W. S. BURNSIDE, M.A.)—Supposing that two quadrics have double contact, their axes remaining parallel; then (1) determine the *complete* locus of the centre of one quadric relative to the other; and (2) find the surfaces traced out by their chords of contact.

I. *Solution by the* Rev. J. WOLSTENHOLME, M.A.

If two conicoids have double contact, they intersect in two plane curves real or impossible. Let these two be

$$ax^2+by^2+cz^2=1, \quad A(x-X)^2+B(y-Y)^2+C(z-Z)^2=1,$$

then $$ax^2+by^2+cz^2-1+\lambda\left\{A(x-X)^2+\ldots-1\right\}$$

must be the product of two linear factors.

Hence $$-\lambda=\frac{a}{A}, \text{ or } \frac{b}{B}, \text{ or } \frac{c}{C}: \text{ suppose } \frac{a}{A} \quad \ldots\ldots\ldots\ldots(1),$$

then $$X=0, \ (b+\lambda B)(c+\lambda C)\left\{\lambda(BY^2+CZ^2-1)-1\right\}$$
$$=\lambda^2C^2Z^2(b+\lambda B)+\lambda^2B^2Y^2(c+\lambda C) \ldots\ldots(2);$$

or $$X=0, \quad \frac{BbY^2}{Ab-Ba}+\frac{CcZ^2}{Ac-Ca}=\frac{1}{A}-\frac{1}{a};$$

and there will be solutions giving two conics in the other principal planes, but these need not all be real.

II. *Solution by the* PROPOSER.

1. Let the equations of the quadrics be written in the forms

$$U \equiv ax^2+by^2+cz^2-1=0,$$
$$V \equiv a'(x-\alpha)^2+b'(y-\beta)^2+c'(z-\gamma)^2-1=0;$$

then, since they have double contact, $V+\lambda U \equiv LM$, whence the discriminant of $V+\lambda U$ vanishes, giving

$$\frac{aa'}{\lambda a+a'}\alpha^2+\frac{bb'}{\lambda b+b'}\beta^2+\frac{cc'}{\lambda c+c'}\gamma^2=1+\frac{1}{\lambda};$$

also $$(\lambda a+a')(\lambda b+b')(\lambda c+c')=0,$$

as the section of $V+\lambda U$ by a plane is two lines.

Taking one of the roots of the latter equation, and substituting it for λ in the former, *sc.* $\lambda=-\frac{c'}{c}$, we find

$$\frac{aa'.\alpha^2}{a'c-ac'}+\frac{bb'.\beta^2}{b'c+bc'}=\frac{1}{c}-\frac{1}{c'} \quad \text{and} \quad \gamma=0$$

as the equations of a conic on which the centre must lie. Similarly we get two other conics (all of which may not be real).

2. It is easily seen that the chord of contact generates cylinders perpendicular to the principal planes; and if the coordinates of the centre be (α, β), the chord of contact passes through the point (x, y) given by the equations $\alpha=(\lambda a'+a)x$, $\beta=(\lambda b'+b)y$, $0=\lambda c'+c$; whence from the three conics we derive three cylinders.

2772. (Proposed by R. TUCKER, M.A.)—The curves

$$y^2 = 4ax \ldots (1), \quad y = a\left(\epsilon^{\frac{x}{2a}} + \epsilon^{-\frac{x}{2a}}\right) \ldots (2), \quad \text{and} \quad \sin\frac{y}{2a} = \epsilon^{\frac{x}{2a}} \ldots (3),$$

are referred to the same origin and axes; show that, at the points for which a tangent to (2) is at right angles to tangents to (1) and (3), the radii of curvature are in continued proportion.

I. *Solution by the* Rev. J. WOLSTENHOLME, M.A.

The intrinsic equations of the three are

$$(1)\ \frac{ds}{d\phi} = \frac{2a}{\sin^2\phi}, \quad (2)\ \frac{ds}{d\phi} = \frac{2a}{\cos^2\phi}, \quad (3)\ \frac{ds}{d\phi} = \frac{2a}{\sin\phi},$$

ϕ being the angle made by the tangent with the axis of x, from which the result is obvious.

II. *Solution by the* PROPOSER.

By reference to my article on Radials (*Reprint*, Vol. I., §§ 10, 13, 22), it will be seen that the radials for (1), (2), (3) are, respectively,

$$r\cos^2\theta = 2a, \quad r\sin^2\theta = 2a, \quad r\cos\theta = 2a.$$

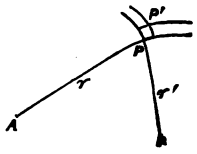
Hence, if ρ_1, ρ_2, ρ_3 be the radii of curvature supposed in the question, we have

$$\rho_1\rho_3 = 4a^2\sec^4\theta = (2a\sec^2\theta)^2 = \rho_2^2,$$

since in (2) we have to write $\frac{1}{2}\pi - \theta$ for θ.

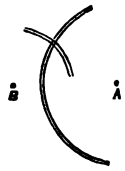
2398. (Proposed by M. W. CROFTON, F.R.S.)—A point P is laid down in a plane by taking random values for AP and BP, its distances from two fixed points A, B; and the plane is covered with an infinite multitude of such points. Show that their distribution will be such that the circle on AB as diameter will be the line of maximum density; and that all circles through A, B are lines of equal density—the density being infinitely small along the line AB.

I. *Solution by the* PROPOSER.



Draw circles from A, B as centres through any point P; radii r, r'; and then two consecutive circles, radii $r+\delta r$, $r'+\delta r'$; δr, $\delta r'$ being constant increments. Then the number of the points which fall inside the elementary parallelogram PP′ is constant. Hence the density at $P \propto \frac{1}{PP'} \propto \frac{\sin P}{\delta r\,\delta r'} \propto \sin P$; so that the density is proportional to sin P. The rest follows.

II. *Solution by the* Rev. J. Wolstenholme, M.A.



If we take a point P situated within the elementary area included within two circles (centre A, radii r, $r+\delta r$), and two circles (centre B, radii r', $r'+\delta r'$); then, δr, $\delta r'$ being by the conditions of the problem independent of the position of P, the density about P will vary inversely as the area included by these four curves (I mean that δr, $\delta r'$ being assigned arbitrary values, the number of points of intersection included within this area will be given, and therefore the density varies inversely as the area). But if we take the equations of the circles $(x+a)^2+y^2=r^2$, $(x-a)^2+y^2=r'^2$, $(\mathrm{AB}\equiv 2a)$, this elementary area is (see Todhunter's *Integral Calculus*, Change of Variables in a Multiple Integral) $\pm\left(\frac{dx}{dr}\frac{dy}{dr'}-\frac{dy}{dr}\frac{dx}{dr'}\right)\delta r\,\delta r'$; or the density varies inversely as $\frac{dx}{dr}\frac{dy}{dr'}-\frac{dy}{dr}\frac{dx}{dr'}$, or varies directly as $\frac{y}{rr'}$. The density will therefore be constant when $\frac{y^2}{(x^2+y^2+a^2)-4a^2x^2}$ or $\frac{y^2}{(x^2+y^2-a^2)+4a^2y^2}$ is constant, and for any circle through AB, $x^2+y^2-a^2=2ky$; therefore the density at any point on such circle is constant, and is to the density on the circle whose diameter is AB as $\frac{1}{\sqrt{(a^2+k^2)}}:\frac{1}{a}$, or as $a:\sqrt{(a^2+k^2)}$; hence the density is a maximum along the circle whose diameter is AB.

2546. (Proposed by R. Tucker, M.A.)—Eliminate θ between

$$\sin\theta\left\{x\sin(60^\circ-\mathrm{A}-\theta)+y\cos(60^\circ-\mathrm{A}-\theta)\right\}+y\sin\mathrm{A}=0,$$

$$\sin(\mathrm{A}+\theta)\left\{y\cos(60^\circ+\theta)-(x-a)\sin(60^\circ+\theta)\right\}=y\sin\mathrm{A}.$$

Solution by the Proposer.

Introducing quantities λ, μ, p, q, &c. to denote expressions independent of θ, we have

$$x\cos(60^\circ-\mathrm{A}-2\theta)-y\sin(60^\circ-\mathrm{A}-2\theta)$$
$$=x\cos(60^\circ-\mathrm{A})-y\sin(60^\circ-\mathrm{A})-2y\sin\mathrm{A}=\lambda,$$

$$(x-a)\cos(60^\circ+\mathrm{A}+2\theta)+y\sin(60^\circ+\mathrm{A}+2\theta)$$
$$=(x-a)\cos(60^\circ-\mathrm{A})+y\sin(60^\circ-\mathrm{A})+2y\sin\mathrm{A}=\mu;$$

whence $\quad p\cos 2\theta+p'\sin 2\theta=\lambda,\quad q\cos 2\theta+q'\sin 2\theta=\mu,$

or $\quad (pq'-p'q)^2=(\lambda q'-\mu p')^2+(\mu p'-\lambda q)^2,$

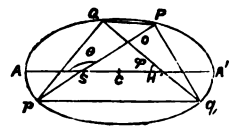
where

$$p=x\cos(60^\circ-\mathrm{A})-y\sin(60^\circ-\mathrm{A}),\quad q=y\sin(60^\circ+\mathrm{A})+(x-a)\cos(60^\circ+\mathrm{A}),$$
$$p'=x\sin(60^\circ-\mathrm{A})+y\cos(60^\circ-\mathrm{A}),\quad q'=y\cos(60^\circ+\mathrm{A})-(x-a)\sin(60^\circ+\mathrm{A}).$$

2526. (Proposed by R. TUCKER, M.A.)—Through the foci of an ellipse two chords are drawn, and with them as diagonals a quadrilateral is formed; find (1) the maximum or minimum figure when they intersect at a constant angle, and (2) the locus of their point of intersection when the quadrilateral is a maximum or a minimum.

Solution by the PROPOSER.

1. Let $PQpq$ be one of the figures whose diagonals intersect in O; then area $= \frac{1}{2}Pp.Qq \sin O$.



If now $ASP = \theta$, $SOH = \alpha$, then area of

$PQpq = 2c^2 + (1 - e^2\cos^2\theta)\left\{1 - e^2\cos^2(\theta - \alpha)\right\}$;

hence for a maximum or minimum,

$$1 - e^2\cos^2\theta - e^2\cos^2(\theta - \alpha) + e^4\cos^2\theta\cos^2(\theta - \alpha) \ldots\ldots\ldots (\alpha)$$

must be a minimum or a maximum;

therefore $0 = \sin 2\theta + \sin 2(\theta - \alpha) - 2e\sin(2\theta - \alpha)\cos\theta\cos(\theta - \alpha)$;

or, $\sin(2\theta - \alpha)\cos\alpha = e^2\sin(2\theta - \alpha)\cos\theta\cos(\theta - \alpha)$;

therefore $2\theta = \alpha$ or $\pi + \alpha$ ..... (i),

and $2\cos\alpha = e^2\left\{\cos(2\theta - \alpha) + \cos\alpha\right\}$, i.e., $\cos(2\theta - \alpha) = \frac{2 - e^2}{e^2}\cos\alpha$...(ii).

Now if u be the variable expression (α),

$$\frac{d^2u}{d\theta^2} = (2 - e^2)\cos\alpha\cos(2\theta - \alpha) - e^2\cos 2(2\theta - \alpha) \ldots\ldots (\beta).$$

If $2\theta = \alpha$, $(\beta) = (2 - e^2)\cos\alpha - e^2$;

if $\cos\alpha > \frac{e^2}{2 - e^2}$, the quadrilateral is a maximum;

if $\cos\alpha < \frac{e^2}{2 - e^2}$, it is a minimum.

If $2\theta = \pi + \alpha$, $(\beta) = -\left\{(2 - e^2)\cos\alpha + e^2\right\}$;

this, if α is acute, gives a minimum value for the quadrilateral; if obtuse, the quadrilateral will still be a minimum until α reaches the value found by making the above vanish, and then it becomes a maximum.

For the value (ii), $(\beta) = e^2 - \frac{(2 - e^2)^2}{e^2}\cos^2\alpha$;

this leads to the same results as above obtained for α, but θ will be different.

2. For the second part of the question we have

$\sin(\theta - \phi) \div \left\{1 - e^2\cos^2\theta - e^2\cos^2\varphi + e^4\cos^2\theta\cos^2\phi\right\} =$ a maximum or minimum;

whence, differentiating with respect to ϕ, we have

$$e^2\sin 2\phi\sin(\theta - \phi) = -\cos(\theta - \phi)(1 - e^2\cos^2\phi),$$

or $\cos(\theta - \phi) = e^2\cos\phi\left\{\cos\theta - \sin\phi\sin(\theta - \phi)\right\}$ ..... (γ).

Let now $\angle HSO = \theta'$ and $SO = \rho$, be the coordinates of O, then

$$\frac{\rho}{2ae} = \frac{\sin\phi}{\sin(\theta'+\phi)}, \text{ and } \cot\phi = \frac{2ae-\rho\cos\theta'}{\rho\sin\theta'} \quad \ldots\ldots\ldots(\delta),$$

and (γ) becomes $\sin\theta' - \cos\theta'\cot\phi = -e^2\cot\phi\left\{\cos\theta' + \frac{2ae\sin^2\phi}{\rho}\right\}$;

therefore $$\cot\phi\cos\theta'(1-e^2) - \sin\theta' = \frac{2ae^2\sin\phi\cos\phi}{\rho}.$$

Substituting from (δ), we have

$$\frac{2ae-\rho\cos\theta'}{\rho\sin\theta'}\cos\theta'(1-e^2) - \sin\theta' = \frac{2ae^2}{\rho}\cdot\frac{(2ae-\rho\cos\theta')\rho\sin\theta'}{(\rho^2-4ae\rho\cos\theta'+4a^2e^2)}.$$

This reduces to (changing to rectangular axes)

$$x^4(1-e^2)+x^2y^2(1-e^2)+y^4-6ae(1-e^2)x^2-6aexy^2+12a^2e^2(1-e^2)x^2 + 4a^2e^2(1+e^2)y^2-8a^3e^3(1-e^2)x = 0.$$

2283. (Proposed by R. TUCKER, M.A.)—The eccentric angles of three points P, Q, R on an ellipse being α, β, γ, find the relations these quantities satisfy when PQ, QR are (i) normals at P, Q, (ii) normals at P, R; and find the area of PQR in the two cases; also the maximum triangles.

Solution by the PROPOSER.

The normals at P, Q, R are given by

$$ax\sec\alpha - by\operatorname{cosec}\alpha = c^2 \quad \ldots\ldots\ldots(1),$$
$$ax\sec\beta - by\operatorname{cosec}\beta = c^2 \quad \ldots\ldots\ldots(2),$$
$$ax\sec\gamma - by\operatorname{cosec}\gamma = c^2 \quad \ldots\ldots\ldots(3);$$

and the chords PQ, QR, RP by

$$\frac{x}{a}\cos\tfrac{1}{2}(\alpha+\beta) + \frac{y}{b}\sin\tfrac{1}{2}(\alpha+\beta) = \cos\tfrac{1}{2}(\alpha-\beta) \quad \ldots\ldots\ldots(4),$$

$$\frac{x}{a}\cos\tfrac{1}{2}(\beta+\gamma) + \frac{y}{b}\sin\tfrac{1}{2}(\beta+\gamma) = \cos\tfrac{1}{2}(\beta-\gamma) \quad \ldots\ldots\ldots(5),$$

$$\frac{x}{a}\cos\tfrac{1}{2}(\gamma+\alpha) + \frac{y}{b}\sin\tfrac{1}{2}(\gamma+\alpha) = \cos\tfrac{1}{2}(\alpha-\gamma) \quad \ldots\ldots\ldots(6).$$

Now, in the first case, (1) must be identical with (4), and (2) with (5); this gives us

$$\left.\begin{aligned} a^2\sec\alpha\cos\tfrac{1}{2}(\alpha-\beta) &= c^2\cos\tfrac{1}{2}(\alpha+\beta)\\ -b^2\operatorname{cosec}\alpha\cos\tfrac{1}{2}(\alpha-\beta) &= c^2\sin\tfrac{1}{2}(\alpha+\beta)\end{aligned}\right\} \quad \ldots\ldots\ldots(A),$$

$$\left.\begin{aligned} a^2\sec\beta\cos\tfrac{1}{2}(\beta-\gamma) &= c^2\cos\tfrac{1}{2}(\beta+\gamma)\\ -b^2\operatorname{cosec}\beta\cos\tfrac{1}{2}(\beta-\gamma) &= c^2\sin\tfrac{1}{2}(\beta+\gamma)\end{aligned}\right\} \quad \ldots\ldots\ldots(B).$$

From (A) and (B) we obtain

$$\tan\alpha\tan\tfrac{1}{2}(\alpha+\beta) = -\frac{b^2}{a^2} = \tan\beta\tan\tfrac{1}{2}(\beta+\gamma) \quad \ldots\ldots\ldots(C).$$

In the second case, we have similarly

$$\tan\alpha\tan\tfrac{1}{2}(\alpha+\beta) = -\frac{b^2}{a^2} = \tan\gamma\tan\tfrac{1}{2}(\gamma+\beta) \quad \ldots\ldots\ldots (D).$$

The equations (C) and (D) give us the required relations.

Again, the area of PQR is

$$2ab\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\alpha).$$

The remainder of the question is solved by taking (7) in connexion with (C) and with (D), and differentiating to get the maximum.

2803. (Proposed by Professor HIRST.)—Through every point A on a conic pass three circles which osculate the conic elsewhere, say in B, C, D. Prove that A, B, C, D lie on the circumference of a circle, and find its envelope.

2833. (Proposed by PATRICK O'CAVANAGH.)—(1.) Through every point A on a central conic pass three circles, which osculate the conic elsewhere, say in B, C, D. Prove that the diameter of the circle passing through A, B, C, D is $= \frac{a^2+3b^2}{2b}$ when greatest, and $\frac{b^2+3a^2}{2b}$ when least. (2.) Prove also that the distance between the centre of this circle, and the centre of the circle osculating the conic at B, C, or D, is when greatest $= \frac{3}{4}\left(\frac{a^2+b^2}{b}\right)$, and when least $= \frac{3}{4}\left(\frac{a^2-b^2}{a}\right)$, where a and b are the semi-axes of the conic.

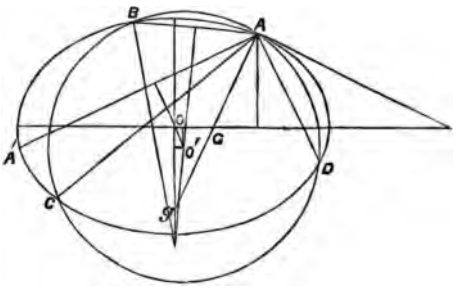
2843. (Proposed by R. TUCKER, M.A.)—Two parabolas are described through the points A, B, C, D of Question 2803, show that their axes intersect on a similar concentric conic, and that their envelopes and the loci of their vertices and foci are quartic curves.

Solution by JAMES DALE.

Let (h, k) be the coordinates of a point A on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and let (x, y) be another point on the ellipse such that the line (h, k) (x, y) makes with the major axis the same angle, but in an opposite direction, as the tangent



at (x, y) does; then the equation to the line (h, k), (x, y), will be

$$k-y = \frac{b^2x}{a^2y}(h-x), \quad \text{or} \quad b^2x^2-a^2y^2-b^2hx+a^2ky = 0 \ldots\ldots\ldots(1)$$

which shows that the points B, C, D such that the chords common to the ellipse and the osculating circles at these points all pass through the same point h, k on the ellipse, all lie on a hyperbola which has its centre at the point $(\frac{1}{2}h, \frac{1}{2}k)$, and its axes parallel to the axes of the ellipse. Since (h, k) is on the ellipse,

$$b^2x^2+a^2y^2-a^2b^2 = 0 \ldots\ldots\ldots\ldots\ldots(2);$$

therefore, adding and subtracting (1) and (2), we find

$$2b^2x^2-b^2hx+a^2ky-a^2b^2 = 0, \quad 2a^2y^2+b^2hx-a^2ky-a^2b^2 = 0 \ldots\ldots(3, 4);$$

showing that A, B, C, D lie also on the parabolas (3) and (4).

Dividing (3) by $2a^2$, (4) by $2b^2$, and adding, we get

$$x^2+y^2-\left(\frac{a^2-b^2}{a^2}\right)\frac{hx}{2}+\left(\frac{a^2-b^2}{b^2}\right)\frac{ky}{2}-\tfrac{1}{2}(a^2+b^2) = 0 \ldots\ldots\ldots(5);$$

or $$\left\{x-\left(\frac{a^2-b^2}{4a^2}\right)h\right\}^2+\left\{y+\left(\frac{a^2-b^2}{4b^2}\right)k\right\}^2 = \tfrac{1}{2}(a+b^2)+\left(\frac{a^2-b^2}{4}\right)^2\left(\frac{h^2}{a^4}+\frac{k^2}{b^4}\right)$$

which shows that the points A, B, C, D lie on a circle, the coordinates of whose centre are $x = \left(\frac{a^2-b^2}{4a^2}\right)h, \quad y = -\left(\frac{a^2-b^2}{4b^2}\right)k,$

$$\text{and} \quad (\text{radius})^2 = \tfrac{1}{2}(a^2+b^2)+\left(\frac{a^2-b^2}{4}\right)^2\left(\frac{h^2}{a^4}+\frac{k^4}{b^4}\right).$$

The position of the centre is readily found by drawing the normal at A, cutting the axes in G, g, respectively; then the coordinates of the centre are respectively $\frac{1}{4}$OG and $\frac{1}{4}$Og.

The centre lies on the ellipse $$\frac{x^2}{\left(\frac{a^2-b^2}{4a}\right)^2}+\frac{y^2}{\left(\frac{a^2-b^2}{4b}\right)^2} = 1.$$

The value of r^2 may be put in the form

$$\tfrac{1}{2}(a^2+b^2)+\left(\frac{a^2-b^2}{4}\right)^2\left\{\frac{1}{b^2}-\frac{h^2}{a^2}\left(\frac{a^2-h^2}{a^2b^2}\right)\right\},$$

showing that r^2 is a maximum or minimum according as h is a minimum or maximum; hence,

for a maximum $h = 0$, $k = b$, and therefore $r = \dfrac{a^2+3b^2}{4b}$;

for a minimum $h = a$, $k = 0$, and therefore $r = \dfrac{3a^2+b^2}{4a}$.

If d be the distance between the centre of this circle, and the centre of the circle osculating the conic at A,

$$d^2 = (a^2-b^2)^2\left\{\frac{h^2}{a^4}\left(\frac{h^2}{a^4}-\frac{1}{4}\right)^2+\frac{k^2}{b^4}\left(\frac{k^2}{b^4}-\frac{1}{4}\right)^2\right\},$$

and, as before,

this is a maximum when $h = 0$, $k = b$, and is then $= \frac{9}{16}\left(\frac{a^2-b^2}{b}\right)^2$,

and a minimum when $h = a$, $k = 0$, and is then $= \frac{9}{16}\left(\frac{a^2-b^2}{a}\right)^2$.

Differentiating the equation to the circle with respect to the variable parameters h, k, we get $\frac{x\,dh}{a^2} - \frac{y\,dk}{b^2} = 0$,

and by the equation to the ellipse, we get $\frac{h\,dh}{a^2} + \frac{k\,dk}{b^2} = 0$;

therefore $kx + hy = 0$; substituting in the equation to the ellipse

$$h = \pm \frac{abx}{(b^2x^2 + a^2x^2)^{\frac{1}{2}}}, \quad k = \mp \frac{aby}{(b^2x^2 + a^2y^2)^{\frac{1}{2}}};$$

substituting these values in the equation to the circle, and reducing, we find the equation to the envelope to be

$$\left\{x^2 + y^2 - \tfrac{1}{2}(a^2 + b^2)\right\}^2 = \left(\frac{a^2 - b^2}{2}\right)^2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right),$$

which represents a bi-circular quartic, having double contact with the ellipse at the extremities of the major and minor axes.

The axes of the parabolas (3) and (4) are given by the equations $x = \frac{1}{2}h$ and $y = \frac{1}{2}k$ respectively, and the intersection of these axes evidently lies on the ellipse $\frac{x^2}{(\frac{1}{2}a)^2} + \frac{y^2}{(\frac{1}{2}b)^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} = 1$, which is similar to, and concentric with, the given ellipse, and of half the linear dimensions.

The envelope of (3) is the quartic $\left(\frac{2x^2 - a^2}{a^2}\right)^2 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 0$, and the envelope of (4) is $\left(\frac{2y^2 - b^2}{b^2}\right)^2 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 0$; which have double contact with the ellipse at the points where it is cut by the lines

$$\frac{x}{a} - \frac{y}{b} = 0 \quad \text{and} \quad \frac{x}{a} + \frac{y}{b} = 0.$$

2852. (Proposed by J. J. Walker, M.A.)—The four points being as in 2803, prove that (1) If a circle be described concentric with the conic, and bisecting the intervals on the axis between the foci, the locus of the centre of the circle ABCD is the conic polar-reciprocal to the given one with respect to the above circle. (2.) That the circle ABCD meets the chord common to the conic, and the osculating circle at A, in points equidistant from the centre of the conic. (3.) That the normal at B, and the line joining the centre of the osculating circle at B with the centre of the circle ABCD, are equally inclined to the axis of the conic.

Solution by James Dale.

1. The polar reciprocal of the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with respect to the circle $x^2 + y^2 = \frac{1}{4}(a^2 - b^2)$, is the conic $\frac{x^2}{\left(\frac{a^2 - b^2}{4a}\right)^2} + \frac{y^2}{\left(\frac{a^2 - b^2}{4b}\right)^2} = 1$, which, as has been already shown, is the locus of the centre of ABCD.

2. The equation to the line joining the centre of the conic to the centre of ABCD is $a^2kx + b^2hy = 0$; and the equation to the chord common to the conic and the osculating circle at A is $(y-k)\,a^2k - (x-h)b^2h = 0$; and these lines being at right angles, it follows that the common chord cuts ABCD in points equidistant from the centre of the conic.

3. AB being the chord common to ABCD and the osculating circle at B, the line joining the centres of these circles will be at right angles to AB, and the normal at B is at right angles to the tangent at B; but AB and the tangent at B are equally inclined to the axis of the conic; therefore, the lines at right angles to them will also be equally inclined to the axis.

2549. (Proposed by J. GRIFFITHS, M.A.) — Solve the simultaneous equations $yz+x = 14,\quad zx+y = 11,\quad xy+z = 10.$

Solution by J. J. WALKER, M.A.

$\left.\begin{aligned} yz+x &= 14 \\ zx+y &= 11 \end{aligned}\right\}$. Multiply the second by z, and subtract, then $x = \dfrac{11z-14}{z^2-1}$.

Similarly, multiplying the first by z, and subtracting, we have

$$y = \frac{14z-11}{z^2-1}, \quad \text{and} \quad xy = \frac{(11z-14)\,(14z-11)}{(z^2-1)^2}.$$

Substituting for xy in the third of the given equations, and reducing, there results $z^5-10z^4-2z^3+174z^2-316z+144 = 0$ .....(1).

Again, adding the first two of the given equations, we obtain

$$x+y = \frac{25}{z+1}, \quad \text{and from the last } xy = 10-z.$$

From these two equations we find

$$2x = \frac{25 \pm \sqrt{\{625+4\,(z-10)\,(z+1)^2\}}}{z+1} = \frac{22z-28}{z^2-1},$$

whence $$\pm\sqrt{\{625+4\,(z-10)\,(z+1)^2\}} = -3\,\frac{z+1}{z-1}.$$

Squaring both sides, and reducing, there results the same equation (1) as above. Trying the factors of 144, we find that neither 2 nor 3 satisfy (1), but 4 does. Hence

$$z = 4, \quad y = \frac{14z-11}{z^2-1} = 3, \quad x = \frac{11z-14}{z^2-1} = 2$$

are three simultaneous values of z, y, x. The biquadratic which gives the four other values of z is $z^4-6z^3-26z^2+70z-36 = 0$, which has two imaginary roots, and two real roots, lying between 1 and 2, 8 and 9, respectively.

2867. (Proposed by S. ROBERTS, M.A.)—Being given four circles, show (1) that their twelve centres of similitude are the points of contact of tangents to a cubic curve drawn from three collinear points upon it; (2) that the curve passes also through the intersections of the straight lines joining the centres of the circles two and two together; and (3) that these results constitute a generalization of the theorem, that, four points being given, the intersections of the straight lines joining them two and two, and the bisecting points of those lines limited by the given points, lie upon the same conic.

Solution by the PROPOSER.

The ordinary geometrical construction for the centres of similitude shows that they lie by threes on sixteen lines, and therefore fulfil the conditions of the points of contact of tangents drawn to a cubic curve from collinear points upon it. Moreover, it will be immediately seen, on constructing the figure, that corresponding external and internal centres of similitude are points, the tangents at which meet on the curve.

If we take any triangle as that of reference, the four corners of a quadrilateral, the sides of which intersect at the vertices of the triangle, are determined by the coordinates (α, β, γ), $(\alpha, \beta, -\gamma)$, $(\alpha, -\beta, \gamma)$, $(-\alpha, \beta, \gamma)$. These do not, however, represent the actual values of the perpendiculars from the four points to the lines of reference. Let k_1, k_2, k_3, k_4 be the multipliers necessary to make the coordinates represent the perpendiculars. They can readily be determined, but it is unnecessary to do this for our purpose. Let r_1, r_2, r_3, r_4 be the radii of the circles drawn about the four points as centres. The coordinates of six centres of similitude (internal) may then be written in the forms

$$\left(\frac{r_2}{k_2}+\frac{r_1}{k_1}\right)\alpha,\quad \left(\frac{r_2}{k_2}+\frac{r_1}{k_1}\right)\beta,\quad \left(\frac{r_2}{k_2}-\frac{r_1}{k_1}\right)\gamma;$$

$$\left(\frac{r_3}{k_3}+\frac{r_1}{k_1}\right)\alpha,\quad \left(\frac{r_3}{k_3}-\frac{r_1}{k_1}\right)\beta,\quad \left(\frac{r_3}{k_3}+\frac{r_1}{k_1}\right)\gamma;$$

$$\left(\frac{r_4}{k_4}-\frac{r_1}{k_1}\right)\alpha,\quad \left(\frac{r_4}{k_4}+\frac{r_1}{k_1}\right)\beta,\quad \left(\frac{r_4}{k_4}+\frac{r_1}{k_4}\right)\gamma;$$

$$\left(\frac{r_3}{k_3}+\frac{r_2}{k_2}\right)\alpha,\quad \left(\frac{r_3}{k_3}-\frac{r_2}{k_2}\right)\beta,\quad \left(\frac{r_2}{k_2}-\frac{r_3}{k_3}\right)\gamma;$$

$$\left(\frac{r_4}{k_3}-\frac{r_2}{k_2}\right)\alpha,\quad \left(\frac{r_4}{k_4}+\frac{r_2}{k_2}\right)\beta,\quad \left(\frac{r_2}{k_2}-\frac{r_4}{k_4}\right)\gamma;$$

$$\left(\frac{r_4}{k_4}-\frac{r_3}{k_3}\right)\alpha,\quad \left(\frac{r_3}{k_3}-\frac{r_4}{k_4}\right)\beta,\quad \left(\frac{r_4}{k_4}+\frac{r_3}{k_3}\right)\gamma;$$

and for the six other centres we have the same values α, β, γ, with reciprocal or inverse coefficients.

If now we suppose the equation of the curve to be formed with $\frac{x}{\alpha}, \frac{y}{\beta}, \frac{z}{\gamma}$ in place of x, y, z, we may write X, Y, Z for these quantities, and take away from the above system α, β, γ. The equation will then be such that the points X', Y', Z', $\frac{1}{X'}, \frac{1}{Y'}, \frac{1}{Z'}$ will be "corresponding points," the tangents thereat meeting on the curve.

Take then the equation

$$aX(Y^2+Z^2)+bY(X^2+Z^2)+cZ(X^2+Y^2)+lXYZ=0,$$

and determine it to be satisfied by the coordinates $(1, 1, K_3)$, $(1, K_2, 1)$, $(K_1, 1, 1)$, K_3 being written for $\left(\frac{r_3}{k_2}-\frac{r_1}{k_1}\right)+\left(\frac{r_2}{k_2}+\frac{r_1}{k_1}\right)$, &c. It will be sufficient to show that the equation will then be also satisfied by another set, say,

$$\frac{r_3}{k_3}+\frac{r_2}{k_2},\ \frac{r_3}{k_3}-\frac{r_2}{k_2},\ \frac{r_2}{k_3}-\frac{r_3}{k_3},\ \text{or}\ \frac{K_2-K_3}{1-K_2K_3},\ 1,\ -1.$$

The rest will follow by symmetry. We must have, then,

$$\begin{vmatrix} \frac{2K_2-K_3}{1-K_2K_3}, & 1+\left(\frac{K_2-K_3}{1-K_2K_3}\right)^2, & -1-\left(\frac{K_2-K_3}{1-K_2K_3}\right)^2, & -\frac{K_2-K_3}{1-K_2K_3} \\ 1+K_3^2\ , & 1+K_3^2\ , & 2K_3\ , & K_3 \\ 1+K_2^2\ , & 2K_2\ , & 1+K_2^2\ , & K_2 \\ 2K_1\ , & 1+K_1^2\ , & 1+K_1^2\ , & K_1 \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} \left(1-\frac{K_2-K_3}{1-K_2K_3}\right)^2, & \left(1-\frac{K_2-K_3}{1-K_2K_3}\right)^2, & 0\ , & -\frac{K_2-K_3}{1-K_2K_3} \\ 0\ , & 0\ , & (1+K_3)^2, & K_3 \\ -(1-K_2)^2\ , & -(1-K_2)^2\ , & (1+K_2)^2, & K_2 \\ (1-K_1)^2\ , & -(1-K_1)^2\ , & 4K_1\ , & K_1 \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} (1-K_2)^2(1+K_3)^2, & (1-K_2)^2(1+K_3)^2, & 0, & K_3(1+K_2)^2-K_2(1+K_3)^2 \\ 0, & 0, & (1+K_3)^2, & K_3 \\ -(1-K_2)^2, & -(1-K_3)^2, & (1+K_2)^2, & K_2 \\ -(1-K_1)^2, & -(1-K_1)^2, & 4K_1\ , & K_1 \end{vmatrix} = 0;$$

and this is identically true, since the first row can be derived from the second and third rows.

The equation of the curve then is of the form, writing L, M, N, P for $X(Y^2+Z^2)$, $Y(X^2+Z^2)$, &c.,

$$\begin{vmatrix} L\ , & M\ , & N\ , & P \\ 1+K_2^2, & 1+K_3^2, & 2K_3\ , & K_3 \\ 1+K_2^2, & 2K_2\ , & 1+K_2^2, & K_2 \\ 2K_1\ , & 1+K_1^2, & 1+K_1^2, & K_1 \end{vmatrix} = 0;$$

or (putting for K_1, K_2, K_3 their values, and reducing),

$$\frac{r_1^2}{k_1^2}(L+M+N+2P)+\frac{r_2^2}{k_2^2}(-L-M+N+2P)$$

$$+\frac{r_3^2}{k_3^2}(-L+M-N+2P)+\frac{r_4^2}{k_4^2}(L-M-N+2P)=0.$$

If the circles are equal, six external centres of similitude lie on the line at infinity and the internal centres of similitude, that is to say, the bisect-

ing points of the sides and diagonals of the quadrilateral formed by the four centres of the circles lie on a conic passing through the vertices of the triangle of reference. If these centres are the centres of the inscribed and escribed circles, the equation is its own simple inverse.

2768. (Proposed by J. J. WALKER, M.A.)—Let $v^3-av^2+bv-c=0$ be one of the five cubics whose roots are simultaneous values of x, y, z, satisfying the three equations in Question 2549 [viz., $yz+x=14$, $zx+y=11$, $xy+z=10$; see *Reprint*, Vol. IX., p. 66, and Vol. XII., p. 40]; show that b must be one of the five roots of the quintic

$$\beta^5-140\beta^4+7608\beta^3-202044\beta^2+2634224\beta-13531024=0,$$

and that the corresponding values of a, c are given by the equations $a+b=35$, $(32-b)c+(36-b)b=404$; also that the five values of c are the roots of the quintic

$$\gamma^5+4\gamma^4-2368\gamma^3+61452\gamma^2-572256\gamma+1783296=0.$$

Solution by the PROPOSER.

For, adding the three given equations, $b+a=35$ .....(1);

again multiplying the same equations, and lastly adding their squares, we find, after substituting for the symmetric functions of x, y, z which enter their values in terms of a, b, c,

$(b-c)^2+(a-1)^2c=1540$... (2), and $a^2-2b+b^2+2(3-a)c=417$... (3).

From (1) and (3),

$$=\frac{404+b(b-36)}{32-b} \text{ [or } (32-b)c+(36-b)b=404] \text{ ..... (4);}$$

and

$$c-b=\frac{404+2b(b-34)}{32-b} \text{ ..... (5).}$$

Eliminating a and c from (1), (2), (4), (5),

$$\left\{\frac{404+2b(b-34)}{32-b}\right\}^2+\frac{(b-34)^2\left\{404+b(b-36)\right\}}{32-b}=1540,$$

or $(2b^2-68b+404)^2+(b-34)^2(b^2-36b+404)(32-b)-1540(b-32)^2$,

or $b^5-140b^4+7608b^3-202044b^2+2634224b-13531024=0$ ..... (6).

This equation is satisfied by $b=26$, whence and from (1) $a=9$, and, from (4), $c=24$. The cubic in v for these values is therefore $v^3-9v^2+26v-24=0$, the roots of which are 2, 3, 4; and these are values of x, y, z which simultaneously satisfy the three equations in Question 2549.

Again, eliminating a between (1) and (2), developing and arranging, there results $(1+c)b^2-70cb+c^2+1156c-1540=0$,

and from (4), $b^2-(36-c)b-32c+404=0$.

Lastly, the coordinates of the foci are

$$\mu = \frac{b^4x'^2 + a^4y'^2 + 8a^4b^2}{8a^2b^2x'}, \quad \nu = \frac{y'}{4}, \quad \text{and} \quad \mu' = \frac{x'}{4}, \quad \nu' = \frac{b^4x'^2 + a^4y'^2 + 8a^2b^4}{8a^2b^2y'},$$

whence for the loci of the foci, after eliminating x', y', we have

$$\left\{16(a^2-b^2)\nu^2 + (8a^2+b^2)b^2\right\}^2 + 16a^2b^2\mu^2\nu^2 - a^2b^4\mu^2 = 0,$$

and

$$\left\{16(b^2-a^2)\mu'^2 + (a^2+8b^2)a^2\right\}^2 + 16a^2b^2\mu'^2\nu'^2 - a^4b^2\nu'^2 = 0.$$

II. *Solution by the* Proposer.

If δ be the eccentric angle for A, then (Salmon's *Conics*, p. 215) B, C, D may be denoted respectively by $-\frac{1}{3}\delta$, $-\frac{1}{3}\delta + 120°$, $-\frac{1}{3}\delta + 240°$; and the equation to any conic through A, B, C, D will be

$$\left(\frac{x}{a}\cos\tfrac{1}{3}\delta + \frac{y}{b}\sin\tfrac{1}{3}\delta - \cos\tfrac{2}{3}\delta\right)\left(\frac{y}{b}\sin\tfrac{1}{3}\delta - \frac{x}{a}\cos\tfrac{1}{3}\delta - \frac{1}{2}\right)$$

$$= k\left\{\frac{x}{a}\cos(60° + \tfrac{1}{3}\delta) + \frac{y}{b}\sin(60° + \tfrac{1}{3}\delta) - \cos(60° - \tfrac{2}{3}\delta)\right\}$$

$$\times \left\{\frac{y}{b}\sin(60° + \tfrac{1}{3}\delta) - \frac{x}{a}\cos(60° + \tfrac{1}{3}\delta) + \frac{1}{2}\right\}.$$

That the curve may represent a parabola gives

$$\left\{k\cos(60° + \tfrac{1}{3}\delta) - \cos^2\tfrac{1}{3}\delta\right\}\left\{\sin^2\tfrac{1}{3}\delta - \mathrm{K}\sin^2(60° + \tfrac{1}{3}\delta)\right\} = 0.$$

Taking the former value, we get

$$\frac{y^2}{b^2} + \frac{x}{2a}\cos\delta - \frac{y}{2b}\sin\delta - \frac{1}{2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1);$$

the other parabola is at once seen to be

$$\frac{x^2}{a^2} - \frac{x}{2a}\cos\delta + \frac{y}{2b}\sin\delta - \frac{1}{2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The respective envelopes are readily seen to be

$$\left(\frac{y^2}{b^2} - \frac{1}{2}\right)^2 = \frac{x^2}{4a^2} + \frac{y^2}{4b^2}, \qquad \left(\frac{x^2}{a^2} - \frac{1}{2}\right)^2 = \frac{x^2}{4a^2} + \frac{y^2}{4b^2}.$$

From (1) and (2) we find the equations to the axes to be

$$4y = b\sin\delta, \quad 4x = a\cos\delta;$$

hence, squaring and adding, we get for locus of intersection

$$16\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 1,$$

a conic similar to and concentric with the given one.

The coordinates of the vertex of (1) are

$$x = a\sec\delta\,(1 + \tfrac{1}{8}\sin^2\delta), \quad y = \tfrac{1}{4}b\sin\delta;$$

hence locus of vertex is $\dfrac{x^2}{a^2}\left(1 - \dfrac{16y^2}{b^2}\right) = \left(1 + \dfrac{2y^2}{b^2}\right)^2$.

Squaring and adding, we have

$$x^2+y^2=\frac{\sqrt{3}}{4}a^2\cos\left(\tfrac{1}{6}\pi+2\theta\right)=\frac{\sqrt{3}}{4}\lambda a^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(\alpha);$$

from (α) $\quad +\sin 6\theta=3\lambda-4\lambda^3, \quad \cos 6\theta=(1-4\lambda^2)\sqrt{(1-\lambda^2)},$

$$\cos 2\theta=\frac{\sqrt{3}.\lambda\pm\sqrt{(1-\lambda^2)}}{2}, \quad \sin 2\theta=\frac{\pm\sqrt{3}\sqrt{(1-\lambda^2)}-\lambda}{2}.$$

Squaring (1), we have

$$x^2+y^2+(y^2-x^2)\cos 6\theta-2xy\sin 6\theta=2a^2\sin^2 2\theta\cos 2\theta,$$

or $\quad x^2+y^2+(y^2-x^2)(1-4\lambda^2)\sqrt{(1-\lambda^2)}-2xy(3\lambda-4\lambda^3)$

$$=a^2(\sqrt{3}.\lambda\pm\sqrt{1-\lambda^2})\frac{3-2\lambda^2\mp2\sqrt{3}.\lambda\sqrt{(1-\lambda^2)}}{4} \quad \ldots\ldots(\beta),$$

in which equation, if we put for λ its value, we have the solution required.

The curve may be traced by its polar equations

$$(1) \ldots\ldots\ldots \rho\sin(3\theta-\phi)=a\sin 2\theta\sqrt{(\cos 2\theta)},$$

$$(\alpha) \ldots\ldots\ldots \rho^2=\frac{\sqrt{3}}{4}a^2\cos\left(\tfrac{1}{6}\pi+2\theta\right);$$

with this substitution, (β) becomes

$$\left[\rho^2\cos 2\phi\left(1-\frac{64\rho^4}{3a^4}\right)\mp\left(\frac{32\rho^4}{3a^2}-\frac{3a^2}{4}\right)\right]\sqrt{\left(1-\frac{16\rho^4}{3a^4}\right)}$$

$$-\rho^2\sin 2\phi\frac{4\rho^2}{\sqrt{3}.a^2}\left(3-\frac{64\rho^4}{3a^2}\right)=0.$$

2930. (Proposed by Professor SYLVESTER.)—Given

$$x+y+z-\xi-\eta-\zeta=i+j+k=i'+j'+k',$$

$$xy+xz+yz-\xi\eta-\xi\zeta-\eta\zeta=i(y+z)+j(z+x)+k(x+y)$$

$$=i'(\eta+\zeta)+j'(\zeta+\xi)+k'(\xi+\eta),$$

$$xyz-\xi\eta\zeta=iyz+jzx-kxy=i'\eta\zeta+j\zeta\xi+k'\xi\eta;$$

prove that

$$\frac{(x-y)^2(x-z)^2(y-z)^2}{(\xi-\eta)^2(\xi-\zeta)^2(\eta-\zeta)^2}=\frac{i'j'k'}{ijk},$$

and that, in like manner, if there be any number of quantities $x, y, z, t \ldots$, and an equal number $\xi, \eta, \zeta, \tau \ldots$, connected together in a manner similar to the above, then

$$\frac{(x-y)^2(x-z)^2(x-t)^2(y-z)^2(y-t)^2(z-t)^2}{(\xi-\eta)^2(\xi-\zeta)^2(\xi-\tau)^2(\eta-\zeta)^2(\eta-\tau)^2(\zeta-\tau)^2}=\frac{i'j'k'l'\ldots}{ijkl\ldots}.$$

Solution by W. H. LAVERTY, B.A.

Let

$$x+y+z=a_1; \quad yz+zx+xy=b_1; \quad xyz=c_1;$$

$$\xi+\eta+\zeta=\alpha_1; \quad \eta\zeta+\zeta\xi+\xi\eta=\beta_1; \quad \xi\eta\zeta=\gamma_1;$$

$$a_1-\alpha_1=\mathrm{A}; \quad b_1-\beta_1=\mathrm{B}; \quad c_1-\gamma_1=\mathrm{C}.$$

Now the condition that $lx+my=p$ may touch the conic $ax^2+2bxy+cy^2+2dx+2ey+f=0$ is the condition that

$$ax^2+2bxy+cy^2+2(dx+ey)\frac{lx+my}{p}+f\left(\frac{lx+my}{p}\right)^2=0$$

may give equal roots in $x:y$,

or $$\left(a+\frac{2dl}{p}+\frac{fl^2}{p^2}\right)\left(c+\frac{2lm}{p}+\frac{fm^2}{p^2}\right)=\left(b+\frac{dm+el}{p}+\frac{flm}{p^2}\right)^2;$$

or $$0=(ac-b^2)p^2+2p\left\{l(cd-be)+m(ae-bd)\right\}+l^2(cf-e^2)+2lm(ed-bf)+m^2af-d^2),$$

an equation whose roots are p_1, p_2; and substituting in (A), we must have the equation

$$\left\{(l\mathrm{X}+m\mathrm{Y})^2-\mu(l^2+m^2)\right\}(ac-b^2)+2(l\mathrm{X}+m\mathrm{Y})\left\{l(cd-be)+m(ae-bd)\right\}+l^2(cf-e^2)+2lm(ed-bf)+m^2(af-d^2)=0$$

true for all values of $l:m$. Whence

$$(\mathrm{X}^2-\mu)(ac-b^2)+2\mathrm{X}(cd-be)+cf-e^2=0,$$
$$(\mathrm{Y}^2-\mu)(ac-b^2)+2\mathrm{Y}(ae-bd)+af-d^2=0,$$
$$\mathrm{XY}(ac-b^2)+\mathrm{X}(ae-bd)+\mathrm{Y}(cd-be)+ed-bf=0.$$

It is obvious that all conics through the four foci of a given conic are rectangular hyperbolas concentric with the given conic; because the four are at the corners of a parallelogram, and each is the centre of perpendiculars of the triangle formed by the other three. [See Salmon's *Conics*, 5th ed., Art. 258, and Wolstenholme's *Book of Mathematical Problems*, Question 690.]

2820. (Proposed by R. Tucker, M.A.)—Find the locus of the intersection of perpendicular normals of a Lemniscate.

Solution by the Proposer.

Taking for equations to the Lemniscate

$$x=a\cos\theta\cos^{\frac{1}{2}}2\theta,\quad y=a\sin\theta\cos^{\frac{1}{2}}2\theta,$$

we have for the normals at θ and ϕ

$$y-x\tan 3\theta=-a\sin 2\theta\cos^{\frac{1}{2}}2\theta\sec 3\theta,$$
$$y-x\tan 3\phi=-a\sin 2\phi\cos^{\frac{1}{2}}2\phi\sec 3\phi,$$

for perpendicularity $\qquad \phi=\frac{1}{3}\pi+\theta;$

and the equations become

$$y\cos 3\theta-x\sin 3\theta=-a\sin 2\theta\cos^{\frac{1}{2}}2\theta \quad\ldots\ldots\ldots\ldots(1),$$
$$y\sin 3\theta+x\cos 3\theta=a\sin(\tfrac{1}{3}\pi+2\theta)\cos^{\frac{1}{2}}(\tfrac{1}{3}\pi+2\theta) \quad\ldots\ldots\ldots(2).$$

putting $a = a' = a'' = \ldots = 1 = b = b' = b'' \ldots$ &c., and is therefore the coefficient of x^r in $(1+p_1x)^{m_1}(1+p_2x)^{m_2}\ldots$, if m_1 of the groups contain the same number of letters, viz. p_1, and so on; and this coefficient is the above-named series, which proves the theorem.

[The particular case in which $p_1 = 1$, $p_2 = 2$, $p_3 \ldots$ &c. $= 0$, has been proved independently by Prince CAMILLE DE POLIGNAC, in a paper read by him before the London Mathematical Society.]

2389. (Proposed by Professor CREMONA.)—Deux droites qui divisent harmoniquement les trois diagonales d'un quadrilatère recontrent en quatre points harmoniques toute conique inscrite dans le quadrilatère.

Solution by the Rev. R. TOWNSEND, F.R.S.

Denoting by M and N any pair of lines dividing harmonically the three diagonals of the quadrilateral; by A and A′, B and B′, C and C′ the three pairs of concurrent lines passing through their six extremities; and by T and T′ the pair of concurrent tangents to any conic inserted in the quadrilateral; then, since, by a well-known property, the four pairs of concurrent lines A and A′, B and B′, C and C′, T and T′ are in involution, and since, by hypothesis, M and N divide harmonically the three angles AA′, BB′, CC′, therefore they divide harmonically the angle TT′; and therefore, &c.

2821. (From WOLSTENHOLME'S *Book of Mathematical Problems.*)—Show that the mean value of the distance from one of the foci of all points within a given prolate spheroid is $\frac{1}{4}a(3+e^2)$, $2a$ being the axis and e the eccentricity.

Solution by R. W. GENESE.

The mean value of the focal distances of all points of the surface from the focus of a prolate spheroid is a: for, clearly, if S and H be the foci, we have average of SP = average of HP over same surface

= average of $\frac{1}{2}(\mathrm{SP}+\mathrm{HP}) = a$.

Let α, β be the axes of a confocal spheroid, whose volume therefore is $\pi\alpha\beta^2$; and $\alpha^2-\beta^2 =$ constant $= c^2$. The volume of a confocal stratum is therefore

$$\frac{d}{d\alpha}\pi\alpha(\alpha^2-c^2) = \pi(3\alpha^2-c^2)\,d\alpha;$$

hence average value of SP $= \dfrac{\int_c^a \alpha\cdot\pi(3\alpha^2-c^2)\,d\alpha}{\pi ab^2}$

$$= \frac{\frac{3}{4}(a^4-c^4)-\frac{1}{2}c^2(a^2-c^2)}{ab^2} = \frac{3(a^2+c^2)-2c^2}{4a}$$

$$= \tfrac{1}{4}a(3+e^2), \text{ since } c = ae.$$

[A Solution by Mr. TUCKER is given on p. 87 of Vol. XI. of the *Reprint.*]

2922. (Proposed by M. W. CROFTON, F.R.S.)—

If $x_1 = \frac{x_0}{1+a_0x_0}$, $x_2 = \frac{x_1}{1+a_1x_1}$, $x_3 = \frac{x_2}{1+a_2x_2}$, &c.,

prove that $x_n = \frac{x_0}{1+hx_0}$, where $h = a_0 + a_1 + \ldots + a_{n-1}$.

Solution by the Rev. R. HARLEY, F.R.S.; A. B. EVANS, M.A.; *and others.*

If $x_p = \frac{x_{p-1}}{1+a_{p-1}x_{p-1}}$, we have

$$a_0 = \frac{1}{x_1} - \frac{1}{x_0}, \quad a_1 = \frac{1}{x_2} - \frac{1}{x_1}, \quad \ldots\ldots\ldots \quad a_{n-1} = \frac{1}{x_n} - \frac{1}{x_{n-1}};$$

therefore $\Sigma a = \frac{1}{x_n} - \frac{1}{x_0}$, or $x_n = \frac{x_0}{1+x_0\Sigma a}$.

So that, whatever a_0, a_1, a_2, &c. may be, the theorem is true.

2615. (Proposed by R. W. SYMES, B.A.)—If

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0$$

be an algebraical equation whose roots are α, β, γ, ..., find the equation whose roots are the different values of

$$\frac{dp_1}{d\alpha} + \frac{dp_2}{d\alpha} + \ldots\ldots + \frac{dp_n}{d\alpha}.$$

Solution by the PROPOSER.

Let $$y = \frac{dp_1}{d\alpha} + \frac{dp_2}{d\alpha} + \ldots\ldots + \frac{dp_n}{d\alpha}.$$

Now, in my solution of Question 2367 (*Reprint*, Vol. IX., p. 69), I have shown that

$$-\frac{dp_1}{d\alpha} = 1, \quad -\frac{dp_2}{d\alpha} = \alpha + p_1, \quad -\frac{dp_3}{d\alpha} = \alpha^2 + p_1\alpha + p_2, \quad \ldots\ldots$$

Therefore the required equation is obtained by eliminating x between the given equation and the following:—

$$y + 1 + (x + p_1) + (x^2 + p_1x + p_2) + \ldots + (x^{n-1} + p_1x^{n-2} + \ldots + p_{n-1}) = 0,$$

or $$y + (1 + x + x^2 + \ldots + x^{n-1}) + p_1(1 + x + x^2 + \ldots + x^{n-2}) + \ldots + p_{n-1} = 0,$$

or $$y + \frac{1-x^n}{1-x} + p_1 \cdot \frac{1-x^{n-1}}{1-x} + p_2 \cdot \frac{1-x^{n-2}}{1-x} + \ldots + p_{n-1} = 0,$$

or $y+\frac{1}{1-x}\left\{(1+p_1+p_2+\ldots+p_n)-(x^n+p_1x^{n-1}+\ldots+p_{n-1}x+p_n)\right\}=0,$

or $y+\frac{a}{1-x}=0$ (where a = sum of the coefficients of the given equation),

or $x=\frac{a+y}{y}$; consequently the required equation is

$$(y+a)^n+p_1(y+a)^{n-1}y+\ldots+p_{n-1}(y+a)y^{n-1}+p_ny^n=0.$$

2790. (Proposed by A. Martin.)—Find the mean distance of all the points in a right circular cylinder from one end of the axis.

Solution by the Proposer.

Let a be the length of the cylinder, r the radius of its base, (x, y) the coordinates of any point within the cylinder, the axis of x coinciding with the axis of the cylinder, and M the mean distance required. Then

$$\mathrm{M}=\frac{\int_0^a\int_0^r 2\pi y\,(x^2+y^2)^{\frac{1}{2}}\,dx\,dy}{\pi ar^2}=\frac{1}{ar^2}\int_0^a\int_0^r y\,(x^2+y^2)^{\frac{1}{2}}\,dx\,dy$$

$$=\frac{2}{3ar^2}+\int_0^a\left\{(x^2+r^2)^{\frac{3}{2}}-x^3\right\}dx$$

$$=\frac{2}{3ar^2}\int_0^a x^2\,(x^2+r^2)^{\frac{1}{2}}\,dx+\frac{2}{3a}\int_0^a(x^2+r^2)^{\frac{1}{2}}\,dx-\frac{1}{3ar^2}\int_0^a x^3\,dx$$

$$=\frac{(2a^2+5r^2)\,(a^2+r^2)^{\frac{1}{2}}}{12r^2}-\frac{a^3}{6r^2}+\frac{r^2}{4a}\log\left(\frac{a+(a^2+r^2)^{\frac{1}{2}}}{r}\right).$$

2932. (Proposed by the Rev. J. Wolstenholme, M.A.)—Given the inscribed and circumscribed circles of a triangle, the envelope of the polar circle is a bicircular quartic.

I. *Solution by* W. K. Clifford, B.A.

Let B, C, X be the inscribed, circumscribed, and polar circles respectively. The circle X has to be such that a triangle self-conjugate with regard to it can be circumscribed to B and inscribed to C; that is, it is subject to a condition of the first and a condition of the second degree in its coefficients. Two such circles can therefore be drawn through an arbitrary point.

Now, *any series of circles, such that two of them can be drawn through an arbitrary point, is one system of bitangent circles of a bicircular quartic.* For

the equation of a circle of the series must contain a variable parameter in the second order; that is, it must be of the form

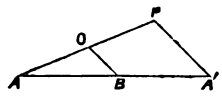
$$*X + 2\theta Y + \theta^2 Z = 0,$$

where $X = 0$, $Y = 0$, $Z = 0$ are circles. But the envelope of this is $XZ = Y^2$, a bicircular quartic.

This important remark is made by CREMONA (*Teoria Geometrica*, ii. 21) in the case of a series of curves of any order and of index 2; that is, such that two of them can be drawn through an arbitrary point. The envelope is always what Prof. CAYLEY (*Edinb. Phil. Trans.* 1868) calls a *trizomal* curve; viz. X, Y, Z being any three curves of the series, its equation may be written

$$\sqrt{\alpha X} + \sqrt{\beta Y} + \sqrt{\gamma Z} = 0.$$

II. *Solution by the* PROPOSER.



Let A, B be the centres of the given circumscribed and inscribed circles, a, b their radii, $AB = \delta \equiv (a^2 - 2ab)^{\frac{1}{2}}$, P the centre of the polar circle of one of the triangles; then if O be middle point of AP, $OB = \frac{1}{2}a - b$, because O is the centre of the nine-point circle of radius $\frac{1}{2}a$ which touches the inscribed circle. Hence, if AB be produced to A', so that $BA' = AB$, A' is a fixed point, and $A'P = a - 2b$, or the locus of P is a circle. Now take A' for origin of polar coordinates, and let ρ be the radius of the polar circle whose centre is P; then $AP^2 = a^2 + 2\rho^2$, whence the equation of the polar circle is, ($\angle PA'A \equiv \alpha$,)

$$\rho^2 = r^2 + (a-2b)^2 - 2r(a-2b)\cos(\theta - \alpha),$$

also

$$a^2 + 2\rho^2 = (a-2b)^2 + 4\delta^2 - 4\delta(a-2b)\cos\alpha;$$

whence the equation of the polar circle, involving only the parameter α, is

$$r^2 - \delta^2 + 2b^2 = 2(a-2b)\left\{r\cos(\theta-\alpha) - \delta\cos\alpha\right\},$$

and for the envelope

$$0 = 2(a-2b)\left\{r\sin(\theta-\alpha) + \delta\sin\alpha\right\};$$

or the equation of the envelope is

$$(r^2 - \delta^2 + 2b^2)^2 = 4(a-2b)^2(r^2 - 2r\delta\cos\theta + \delta^2);$$

or, in rectangular coordinates, A' origin and A'A axis of x, is

$$(x^2 + y^2 - \delta^2 + 2b^2)^2 = 4(a-2b)^2\left\{(x-\delta)^2 + y^2\right\},$$

a bicircular quartic, in fact a Cartesian, of which B is one focus.

2623. (Proposed by W. S. M'CAY, B.A.)—If a conic pass through the four points of contact of tangents to a cubic from a point (A) on the curve, and through two other points (B, C) on the cubic; then A is the pole of BC with regard to the conic.

* [Mr. WOLSTENHOLME remarks that "it is not necessary that X, Y, Z should be all circles; it is sufficient that one be a circle and the others straight lines. Thus the envelope might, so far as depends on this reasoning, be a circular cubic."]

I. *Solution by the* Rev. R. Townsend, F.R.S.

This property is evidently the converse of the following:—"If a variable conic pass through four fixed points, the locus of the points of contact of tangents to it from a fifth fixed point is a cubic, passing through the given points, and touching at the four the four lines connecting them with the fifth," which may be readily proved as follows:—

The equation of the variable conic being $U + kV = 0$, where U and V are any two fixed conics passing through the four points, and k the variable parameter; if $(\alpha', \beta', \gamma')$ be the coordinates of the fifth point, and Δ the symbol of operation $\alpha' \frac{d}{d\alpha} + \beta' \frac{d}{d\beta} + \gamma' \frac{d}{d\gamma}$; then, since at the points of contact of the two tangents from $(\alpha', \beta', \gamma')$ to $U + kV$ we have $\Delta U + k \cdot \Delta V = 0$; therefore, eliminating k, we have $U \cdot \Delta V - V \cdot \Delta U = 0$, and therefore &c. as regards the locus being a cubic passing through the four points. And since, at the points of contact of the four tangents to it from $\alpha'\beta'\gamma'$, we have $U \cdot \Delta^2 V - V \cdot \Delta^2 U = 0$, therefore &c., as regards its touching the four lines. That it passes through $(\alpha', \beta', \gamma')$ is evident *à priori*, a conic of the variable system passing through it.

N.B.—The corresponding property in Geometry of Three Dimensions, —viz., "If a variable quadric pass through a fixed quartic, the locus of the points of contact of tangents to it from a fixed point is a cubic, passing through the quartic and point, and touching along the former the cone connecting it with the latter,"—may evidently be proved in precisely the same manner.

II. *Solution by the* Proposer.

Let U, V be two conics, P, Q the polars of a fixed point A with regard to them; then the locus of the points of contact of tangents from A to the system of conics through the intersections of U, V is $UQ - PV = 0$, a cubic.

But the lines joining A to these intersections touch this cubic; for their equation is $(UV' - VU')^2 = 4\,(UQ - PV)\,(PV' - QU')$

(the dotted letters being the result of substituting the coordinates of A), a locus evidently touching the cubic at the four points.

[A Solution by Mr. Thomson is given on p. 68 of Vol. XI. of the *Reprint*.]

2917. (Proposed by the Rev. J. Wolstenholme, M.A.)—If a conic pass through two given points and touch a given conic at a given point, the chord of intersection with the given conic will pass through a fixed point on the given straight line.

I. *Solution by* T. Cotterill, M.A.

More generally, if a, b are two points on a given conic, and c, d two other points not on the conic, then the conic and straight line through c, d form a cubic. Any conic through a, b, c, d cuts the given conic again in two points which are collinear with a fixed point on the line through c, d; viz., the *opposite* of the four points a, b, c, d on the cubic.

II. *Solution by* F. D. THOMSON, M.A.; J. DALE; *and others.*

If a circle cut a conic in four points, the pairs of common chords are known to be equally inclined to the major axis of the conic, since the rectangles contained by their segments are equal, and these rectangles are proportional to the parallel diameters.

Hence, when two of the four points coincide, the common chord to the conic and the circle is equally inclined with the common tangent to the major axis.

Therefore the common chord made by a variable circle touching the conic at a given point is parallel to a fixed direction; hence generally, replacing the circular points at infinity by any two fixed points, we see that the common chord in the proposed question passes through a fixed point on the line joining the two given points.

[Mr. TUCKER remarks that the theorem is a particular case of the theorem, "Given four points on a conic, its chord of intersection with a fixed conic passing through two of these points will pass through a fixed point," which is readily seen to lie on the fixed line. Both results follow from Salmon's *Conics*, § 266.]

2907. (Proposed by M. W. CROFTON, F.R.S.)—Given that three events A, B, C occurred in a certain century, A having preceded B, and B having preceded C, this being all the information obtainable, show that the probability (p) that the date of the event B was not distant more than n years from the middle of the century is

$$p = 3\frac{n}{100} - 4\left(\frac{n}{100}\right)^3.$$

I. *Solution by* STEPHEN WATSON.

The total number of years on which A can fall (n being constant) is $50+n$, and the like number for C is $50-n$; hence the required chance is

$$p = \frac{\int_{-n}^{n} (50^2-n^2)\,dn}{\int_{-50}^{50} (50^2-n^2)\,dn} = \frac{50n-\frac{1}{3}n^3}{50^3-\frac{1}{3}50^3} = \frac{3n}{100} - 4\left(\frac{n}{100}\right)^3.$$

II. *Solution by the* REV. J. WOLSTENHOLME, M.A.

If x, $x+y$, $x+y+z$ be the dates (in the century) of the three events, then all that is known is that x, y, z are all positive, and $x+y+z < 100$. The whole possible cases may then be represented by the positions of points lying within the tetrahedron formed by the coordinate planes and the plane $x+y+z=100$; and the favourable cases will correspond to points lying be-

tween the two planes $x+y=50+n$, $x+y=50-n$. The chance will then be

$$\int_{50-n}^{50+n} u(100-u)\,du + \int_{0}^{100} u(100-u)\,du,$$

or $$\int_{1-m}^{1+m} u(2-u)\,du + \int_{0}^{2} u(2-u)\,du, \quad \text{if } m \equiv \tfrac{1}{50}n;$$

or $$\tfrac{2}{3}m - \tfrac{1}{2}m^3, \text{ which } = \tfrac{3}{100}n - 4\left(\tfrac{1}{100}n\right)^3.$$

If $\frac{1}{100}n = \sin\alpha$, the required chance is $\sin 3\alpha$.

2950. (Proposed by Professor SYLVESTER.)—If j be any positive integer, then

$$\frac{j+1}{1.j} + \frac{j+1}{2(j-1)} + \frac{j+1}{3(j-2)} + \ldots\ldots + \frac{j+1}{j.1} = 2\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{j}\right).$$

Solution by the Rev. R. HARLEY, F.R.S.

We have $$\frac{j+1}{1.j} = 1 + \frac{1}{j}, \quad \frac{j+1}{2(j-1)} = \frac{1}{2} + \frac{1}{j-1},$$

$$\frac{j+1}{3(j-2)} = \frac{1}{3} + \frac{1}{j-2}, \quad \ldots\ldots\ldots \quad \frac{j+1}{j.1} = \frac{1}{j} + \frac{1}{1};$$

and, since $$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{j} = \frac{1}{j} + \frac{1}{j-1} + \frac{1}{j-2} + \ldots\ldots + \frac{1}{1},$$

the truth of the theorem is manifest.

Mr. TUCKER and others give the proof thus:—

Assume $$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{j-1} + \frac{1}{j} = S,$$

then $$\frac{1}{j} + \frac{1}{j-1} + \frac{1}{j-2} + \ldots\ldots + \frac{1}{2} + \frac{1}{1} = S;$$

adding, we have

$$\frac{j+1}{1.j} + \frac{j+1}{2(j-1)} + \frac{j+1}{3(j-2)} + \ldots\ldots + \frac{j+1}{j.1} = 2S = 2\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{j}\right).$$

2923. (Proposed by S. ROBERTS, M.A.)—In a bicircular quartic, the points of contact of the four single tangents drawn from the centre of a circle on which four foci lie, are on the circle, and the corresponding points of contact of double tangents also lie on a circle.

Solution by W. K. CLIFFORD, B.A.

A bicircular quartic is its own inverse with regard to any focal circle (Moutard, *Nouvelles Annales*, 1866). The bitangent circles divide themselves into four systems, all the circles of any one system being cut orthogonally by the corresponding focal circle. Through any point of the plane can be drawn two bitangent circles of each system. The two bitangent circles, then, that can be drawn through the centre of the focal circle of their system, are in fact straight lines touching the curve in two pairs of inverse points, which consequently lie on a circle.

The corresponding theory in anallagmatic surfaces is that the centre of each one of the five principal spheres is vertex of a quadric cone doubly tangent to the surface; the curve of contact being the intersection of this cone with a sphere. These five cones noticed by Moutard are independently arrived at by Kummer (*Berlin. Monatsber.*) in the case of the general quartic surface with a nodal conic.

2603. (Proposed by T. COTTERILL, M.A.)—In a plane, (1) on every line, there are two points harmonic conjugates to the conics through four points A, B, C, D. They are real, if the product of the four triangles ABC, BCD, CDA, DAB, and the perpendiculars from the points A, B, C, D on the line, is positive. If a line does not cut a conic, why are the conjugates on the line real, to any set of four points on the conic? Give a rule for determining the nature of the points on a line, when the product of the triangles is (i) positive, (ii) negative.

(2.) Through any point, there are two lines conjugate to the conics inscribed in four lines. They are real in those regions of the plane formed by the four lines in which the product of perpendiculars from the point has the same sign as that of points in the convex quadrilateral. Hence every point on or within a conic inscribed in the four lines lies in these regions.

Solution by W. H. LAVERTY, B.A.

1. The line always cuts the conic and sides of the quadrilateral in points in involution, the double points (real or imaginary) of which are harmonic conjugates to the conics. Now if AA_1, BB_1, CC_1, DD_1 be the perpendiculars respectively from A, B, C, D on BC, DA, DA, BC, and AA_2, BB_2, CC_2, DD_2 those on the given line; then the product of the triangles is of the same sign as $AA_1 . BC . DD_1 . BC . CC_1 . DA . BB_1 . DA$, that is, of the same sign as $AA_1 . BB_1 . CC_1 . DD_1$. But

$AA_1 . BB_1 . CC_1 . DD_1$ is positive for the Euclidean skew quadrilateral ... (1),
and negative for the reentrant quadrilateral .....(2);

and according as $AA_2 . BB_2 . CC_2 . DD_2$ is positive or negative,
so in (1) the system is non-overlapping or overlapping,
and in (2) the system is overlapping or non-overlapping.

Consequently the system is non-overlapping or overlapping, *i. e.*, has real or imaginary double points, according as the two products have the same or opposite signs, *i. e.*, according as the given product of triangles and perpendiculars is positive or negative.

If the line does not cut the conic, the two points where the line meets the conic are imaginary, and the double points which are harmonic conjugates of these are real.

A simple analytical proof of this is as follows:—The equations to the conic and line may be put into the forms

$$\beta^2 + C\gamma^2 + 2k\beta\gamma + 2E\gamma\alpha + 2F\alpha\beta = 0, \quad \text{and} \quad \alpha = 0,$$

where k is indeterminate. For the points where α cuts the conic we have

$$\beta^2 + 2k\beta\gamma + C\gamma^2 = 0; \quad \text{therefore} \quad \beta = \gamma\left\{-k \pm (k^2 - C)^{\frac{1}{2}}\right\},$$

which can only be imaginary when C is positive. But for the double rays we have

$$k = \pm C^{\frac{1}{2}}; \quad \text{therefore} \quad \beta = C^{\frac{1}{2}}\gamma, \quad \beta = -C^{\frac{1}{2}}\gamma,$$

which are always possible when C is positive, that is, when the line does not meet the conic.

2. The first part of this is the exact correlative of the first part of (1). Lines drawn from a point to the vertices of a quadrilateral form an involution of which the tangents to the conic are a pair; and the double rays are conjugate lines of the conic.

For any point inside the convex quadrilateral, the involution of lines is non-overlapping, and therefore the double lines are real. And it is easily seen that, if we move into those regions where we alter the sign of only one perpendicular, the involution becomes overlapping and the double rays imaginary. Also, for any point inside the conic, the double rays are real, as in (1); and as for a point on the conic, the tangent is a double ray, they are real there also. Therefore any point on or within the conic must lie in the stated regions.

[A Solution by Mr. WOLSTENHOLME is given on p. 28 of this volume of the *Reprint*.]

2399. (Proposed by the Rev. R. TOWNSEND, F.R.S.)—Find a point on a sphere such that the triangle determined by the middle points of the three arcs connecting it with three given points on the sphere will have two of its sides given.

Solution by the PROPOSER.

If A, B, C be the three given points, O the required point, and X, Y, Z the three middle points of the three arcs OA, OB, OC; then, since, by a well-known property,

$$\cos YZ = \cos \tfrac{1}{2}BC \,.\, \cos \tfrac{1}{2} \text{ area } BOC,$$
$$\cos ZX = \cos \tfrac{1}{2}CA \,.\, \cos \tfrac{1}{2} \text{ area } COA,$$
$$\cos XY = \cos \tfrac{1}{2}AB \,.\, \cos \tfrac{1}{2} \text{ area } AOB,$$

therefore, if two of the three arcs YZ, ZX, XY be given, the corresponding two of the three areas BOC, COA, AOB are given, and therefore the point O is given as the intersection of two well-known circles.

2914. (Proposed by J. J. WALKER, M.A.)—If ABC be a triangle inscribed in, and concentroidal with, an ellipse whose semi-axes are a, b; prove that (1) the sum of the squares of the sides of such a triangle is constant, and equal to $\frac{9}{2}(a^2+b^2)$; (2) the sum of the squares of the lines joining A, B, C with the fourth point D, in which the circumscribing circle cuts the ellipse, is equal to $3\left\{\frac{1}{2}(a^2+b^2)+r^2\right\}$, r being the semi-diameter drawn to the point; and (3) the sum of the squares of the diameters of the ellipse passing through the points A, B, C, is equal to the sum of the squares of three conjugate diameters.

I. *Solution by* JAMES DALE, *and others.*

Since the diameters AA′, BB′, CC′ respectively bisect BC, CA, AB; Oa, Ob, Oc, which are parallel to BC, CA, AB, are conjugate to OA, OB, OC respectively. By the property of the centre of gravity, $BC^2+CA^2+AB^2=3(OA^2+OB^2+OC^2)$; and since BC passes through the middle point of OA′, and is parallel to OA, $BC^2=3Oa^2$; similarly, $CA^2=3Ob^2$, $AB^2=3Oc^2$;

$$\therefore\ 2(BC^2+CA^2+AB^2)=3\left\{(OA^2+Oa^2)+(OB^2+Ob^2)+(OC^2+Oc^2)\right\}$$
$$=9(a^2+b^2),$$

and
$$OA^2+OB^2+OC^2=Oa^2+Ob^2+Oc^2,$$

which proves (1) and (3). To prove (2), let (x_1, y_1), (x_2, y_2), (x_3, y_3), (h, k) be the coordinates of A, B, C, D; then

$$AD^2=(h-x_1)^2+(k-y_1)^2=r^2+OA^2-2hx_1-2ky_1;$$

so also $BD^2=r^2+OB^2-2hx_2-2ky_2$, $CD^2=r^2+OC^2-2hx_3-2ky_3$;

therefore $AD^2+BD^2+CD^2=3r^2+OA^2+OB^2+OC^2=3\left\{\frac{1}{2}(a^2+b^2)+r^2\right\}$,

because
$$x_1+x_2+x_3=0,\quad y_1+y_2+y_3=0.$$

II. *Solution by* STEPHEN WATSON.

Put a', b' for the semi-axes of the given ellipse; then, taking BC, BA as axes of reference, the equation of the ellipse concentroidal with the triangle is
$$c^2x^2+caxy+a^2y^2-ca(cx+ay)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(\alpha),$$

its centre being at the point $O=(\frac{1}{2}a, \frac{1}{2}c)$, and a line through O parallel to the tangent to (α) at the origin is

$$y-\tfrac{1}{2}c=-\frac{c}{a}(x-\tfrac{1}{2}a).$$

This meets (1) in the point $B'=\left\{\frac{1}{2}a(1+\sqrt{3}),\ \frac{1}{2}c(1+\sqrt{3})\right\}$;

hence
$$OB'^2=\tfrac{1}{4}a^2+\tfrac{1}{4}c^2-\tfrac{1}{4}ac\cos B=\tfrac{1}{4}b^2,$$

also
$$OB^2=\tfrac{1}{9}a^2+\tfrac{1}{9}c^2+\tfrac{2}{9}ac\cos B=\tfrac{2}{9}a^2+\tfrac{2}{9}c^2-\tfrac{1}{9}b^2;$$

therefore $a'^2+b'^2=OB^2+OB'^2=\frac{2}{9}(a^2+b^2+c^2)$, which proves (1).

Denote the point D by (x_1, y_1); then

$$AD^2 = x_1^2 + (y_1 - c)^2 + 2x_1(y_1 - c)\cos B,$$
$$BD^2 = x_1^2 + y_1^2 + 2x_1 y_1 \cos B,$$
$$CD^2 = (x_1 - a)^2 + y_1^2 + 2(x_1 - a)y_1 \cos B,$$
$$r^2 = (x_1 - \tfrac{1}{3}a)^2 + (y_1 - \tfrac{1}{3}c)^2 + 2(x_1 - \tfrac{1}{3}a)(y_1 - \tfrac{1}{3}c)\cos$$

therefore $AD^2 + BD^2 + CD^2 - 3r^2 = \frac{2}{3}(a^2 + c^2 - ac\cos B)$
$= \frac{1}{3}(a^2 + b^2 + c^2) = \frac{2}{3}(a'^2 + b'^2)$,

which proves (2).

Again, OB′ is the semi-conjugate to OB, and let OA′, OC′ b
OA, OC; then, as OB'^2, OB^2 were found above, we have

$OA'^2 = \frac{1}{3}a^2$, $OC'^2 = \frac{1}{3}c^2$, $OA^2 = \frac{2}{3}(b^2 + c^2) - \frac{1}{3}a^2$, $OC^2 = \frac{2}{3}(a^2 +$

therefore $OA'^2 + OB'^2 + OC'^2 = \frac{1}{3}(a^2 + b^2 + c^2) = OA^2 + OB^2 + OC^2$,

which proves (3).

III. *Solution by the* PROPOSER.

1. The coordinates of A, B, C are the roots of the equations

$$4x^3 - 3a^2x - a^2x' = 0, \quad 4y^3 - 3b^2y - b^2y' = 0 \ldots\ldots$$

(x', y') being the fourth point in which the circle circumscrib
triangle ABC meets the ellipse. Now the sum of the squares of t
ferences of the roots of (α) is equal to $\frac{2}{3}a^2$; and the same functio
roots of (β) is equal to $\frac{2}{3}b^2$; whence $AB^2 + BC^2 + CA^2 = \frac{2}{3}(a^2 + b^2)$.

2. Again, if $\alpha, \alpha', \alpha''$ be the roots of (α), and β, β', β'' those of (β),

$$\Sigma(x' - \alpha)^2 = 3x'^2 - 2x'\Sigma\alpha + \Sigma\alpha^2 = 3\left(\tfrac{1}{3}a^2 + x'^2\right);$$

similarly $$\Sigma(y' - \alpha)^2 = 3\left(\tfrac{1}{3}b^2 + x'^2\right);$$

whence $$AD^2 + BD^2 + CD^2 = 3\left\{\tfrac{1}{3}(a^2 + b^2) + r^2\right\}.$$

3. If O be the centre of the ellipse,

$$OA^2 + OB^2 + OC^2 = \Sigma\alpha^2 + \Sigma\beta^2 = \tfrac{2}{3}(a^2 + b^2);$$

but the six squares of these semi-diameters and the conjugates $= 3(a^2 + b^2$

2913. (Proposed by R. TUCKER, M.A.)—Straight lines are drawn from the angles A, B, C of the triangle ABC, through a point O within it, meeting the opposite sides in D, E, F; show that the locus of O, when the perpendiculars from A, B, C upon EF, FD, DE meet in a point, is a cubic through A, B, C. Find also the locus of the latter point.

I. *Solution by* STEPHEN WATSON.

Let the equations of AD, BE, CF be $l\alpha = m\beta = n\gamma$ .....(1);

then that of EF is $l\alpha + m\beta - n\gamma = 0$, and a perpendicular to this from A is

$$\frac{\beta}{\gamma} = \frac{m + l\cos C - n\cos A}{n - m\cos A + l\cos B} \quad \ldots\ldots(2).$$

Similarly,

$$\frac{\gamma}{\alpha} = \frac{n + m\cos A - l\cos B}{l - n\cos B + m\cos C}, \quad \text{and} \quad \frac{\alpha}{\beta} = \frac{l + n\cos B - m\cos C}{m - l\cos C + n\cos A} \quad \ldots\ldots\ldots(3, 4)$$

are the equations of perpendiculars from B and C to FD, DE. When (2), (3), (4) meet in a point, the product of the right-hand members must $= 1$; and eliminating from the result thus obtained the quantities l, m, n, by means of (1), we get, after reduction, for the locus of O the equation

$$\sin^2 A\,(\beta\cos B - \gamma\cos C)\,\alpha^2 + \sin^2 B\,(\gamma\cos C - \alpha\cos A)\,\beta^2 + \sin^2 C\,(\alpha\cos A - \beta\cos B)\,\gamma^2 = 0 \quad \ldots\ldots\ldots\ldots (5).$$

The equations (2), (3), (4) may be written

$$(\beta + \gamma\cos A)\,n - (\gamma + \beta\cos A)\,m + (\beta\cos B - \gamma\cos C)\,l = 0 \quad \ldots\ldots(6),$$

$$-(\alpha + \gamma\cos B)\,n + (\gamma\cos C - \alpha\cos A)\,m + (\gamma + \alpha\cos B)\,l = 0 \quad \ldots\ldots(7),$$

$$(\alpha\cos A - \beta\cos B)\,n + (\alpha + \beta\cos C)\,m - (\beta + \alpha\cos C)\,l = 0 \quad \ldots\ldots(8),$$

and $\alpha\,(6) + \beta\,(7) - \gamma\,(8)$ gives $\dfrac{l}{m} = \dfrac{(\gamma + \beta\cos A)\,\alpha}{(\gamma + \alpha\cos B)\,\beta}$.

Similarly, $\dfrac{m}{n} = \dfrac{(\alpha + \gamma\cos B)\,\beta}{(\alpha + \beta\cos C)\,\gamma}$, $\dfrac{n}{l} = \dfrac{(\beta + \alpha\cos C)\,\gamma}{(\beta + \gamma\cos A)\,\alpha}$;

hence, eliminating l, m, n, we have, from the locus of the "latter point," the equation

$$(\cos A - \cos B\cos C)\,(\beta^2 - \gamma^2)\,\alpha + (\cos B - \cos C\cos A)\,(\gamma^2 - \alpha^2)\,\beta + (\cos C - \cos A\cos B)\,(\alpha^2 - \beta^2)\,\gamma = 0 \quad \ldots\ldots\ldots\ldots(9).$$

Putting successively α, β, $\gamma = 0$, in the cubics (5) and (9), the results give respectively

$$\frac{\sin^2 A}{\cos A}\alpha = \frac{\sin^2 B}{\cos B}\beta = \frac{\sin^2 C}{\cos C}\gamma,$$

$$\frac{\alpha}{\cos A - \cos B\cos C} = \frac{\beta}{\cos B - \cos C\cos A} = \frac{\gamma}{\cos C - \cos A\cos B};$$

hence, in each case, the cubic meets the sides of the triangle in three points such that, when joined to the opposite angles, the three joining lines are concurrent.

II. *Solution by* James Dale, *and others.*

Let AO, BO, CO meet the opposite sides in D, E, F respectively; then, if f, g, h are the coordinates (trilineal) of O, the equations of EF, FD, DE are

$$-\frac{x}{f} + \frac{y}{g} + \frac{z}{h} = 0, \quad \frac{x}{f} - \frac{y}{g} + \frac{z}{h} = 0, \quad \frac{x}{f} + \frac{y}{g} - \frac{z}{h} = 0;$$

and the equations to the perpendiculars from A, B, C are

$$(gh\cos B - hf\cos A + fg)\,y - (gh\cos C + hf - fg\cos A)\,z = 0,$$

$$(gh + hf\cos C - fg\cos B)\,z - (-gh\cos B + hf\cos A + fg)\,x = 0,$$

$$(-gh\cos C + hf + fg\cos A)\,x - (gh - hf\cos C + fg\cos B)\,y = 0.$$

The condition that these lines should pass through one point is

$$(-gh\cos C + hf + fg\cos A)(gh\cos B - hf\cos A + fg)(gh + hf\cos C - fg\cos B)$$
$$-(-gh\cos B + hf\cos A + fg)(gh - hf\cos C + fg\cos B)(gh\cos C + hf - fg\cos A)$$
$$= 0,$$

which, being simplified, shows that (f, g, h) lies on the cubic

$$x\cos A(y^2\sin^2 B - z^2\sin^2 C) + y\cos B(z^2\sin^2 C - x^2\sin^2 A)$$
$$+ z\cos C(x^2\sin^2 A - y^2\sin^2 B) = 0,$$

or $a^2x^2(y\cos B - z\cos C) + b^2y^2(z\cos C - x\cos A) + c^2z^2(x\cos A - y\cos B) = 0.$

This cubic passes through the angular points of the triangle, through the intersection of the perpendiculars, and also through the four points

$$\pm ax = \pm by = \pm bz.$$

Eliminating f, g, h from the equations to the perpendiculars, we find the locus of the intersection of these perpendiculars

$$x(y^2 - z^2)(\cos A - \cos B\cos C) + y(z^2 - x^2)(\cos B - \cos C\cos A)$$
$$+ z(x^2 - y^2)(\cos C - \cos A\cos B) = 0.$$

This cubic also passes through the angular points of the triangle, through the centre of the circumscribing circle, and through the centres of inscribed and escribed circles, and through the intersection of the perpendiculars.

2924. (Proposed by the Rev. J. Wolstenholme, M.A.)—On a focal chord PSQ of a parabola are taken p, q, on opposite sides of S, such that $Sp \cdot Sq = SP \cdot SQ$, and any parabola is described through p, q, and having its axis parallel to that of the former: prove that their chord of intersection will pass through S.

I. *Solution by* W. K. Clifford, B.A.

I consider the following more general question:—

Through a point a let a line B be drawn meeting a conic in l, m; then the quantity

$$al \cdot am \sin BP \cdot \sin BQ \quad \ldots\ldots\ldots\ldots (1)$$

(where P and Q are the asymptotes) is independent of the position of the line B, and may be called the *distance* of the point a from the conic. What now is the locus of a point equidistant from two given conics?

Let $C_2 = 0$, $D_2 = 0$ be the equations of the conics, and let a^2C_2 denote the result of substituting the coordinates of the point a for the variables in C_2; also let i, j be the circular points at infinity. Then I find that the distance (above defined) of the point a from C_2 is

$$\frac{a^2C_2}{(aij)^2 \cdot \sqrt{(i^2C_2 \cdot j^2C_2)}} \quad \ldots\ldots\ldots\ldots (2),$$

where, of course, (aij) means the determinant formed with the coordinates of the points a, i, j.

This being so, the equation of the required locus is

$$\frac{C_2}{\sqrt{(i^2C_2 \cdot j^2C_2)}} = \frac{D_2}{\sqrt{(i^2D_2 \cdot i^2D_2)}} \quad \ldots\ldots\ldots\ldots (3),$$

showing that the locus is a conic passing through the intersections of the two given ones.

Now if we are using Cartesian coordinates, and if the two conics are similar and similarly situated, it is easy to see that the terms of the second order have entirely disappeared from the equation (3); which indicates that the line at infinity is part of the locus. The remainder of it is then their finite chord of intersection; which is a true radical axis, in the sense that if any line whatever is met by the radical axis in a, by C_2 in l, m, and by D_2 in l', m', we must have always

$$al \,.\, am = al' \,.\, am' \ldots\ldots\ldots\

the form of which shows that it touches the sides of the triangle of reference in the points

$$(x=0,\ my-nz),\quad (y=0,\ nz-lx=0),\quad (z=0,\ lx-my=0),$$

which are the harmonic conjugates of L, M, N respectively.

The tangents to the Hessian at L, M, N are respectively

$$4mn\,(lx+my+nz)-kx=0\,;\qquad 4nl\,(lx+my+nz)-ky=0\,;$$
$$4lm\,(lx+my+nz)-kz=0\,;$$

and hence are obtained the equations of PP′, QQ′, RR′, viz. $my-nz=0$, $nz-lx=0$, $lx-my=0$, intersecting in $lx=my=nz$, which is also the point of intersection of Pp, Qq, Rr.

II. *Solution by the* Proposer.

1. It may be shown that, if the equation to a curve be given under the form $U\equiv\phi(x, y, z, t)=0$, where x, y, z, t are connected by the identical relation

$$ax+by+cz+dt=0 \qquad \text{.....(1),}$$

that the polar conic of a point (x, y, z, t) on the curve has for its equation

$$S\equiv(A, B, C, D, E, F, G, H, K, L)(x, y, z, t)^2=0,$$

where $A=\dfrac{d^2U}{dx^2}$, ... $E=\dfrac{d^2U}{dy\,dz}$, ... $H=\dfrac{d^2U}{dx\,dt}$, ...

Now for a point of inflexion S becomes two straight lines, and it may be shown that the discriminant of S is given by

$$\Delta\equiv\begin{vmatrix} A, & G, & F, & H, & a\\ G, & B, & E, & K, & b\\ F, & E, & C, & L, & c\\ H, & K, & L, & D, & d\\ a, & b, & c, & d, & 0 \end{vmatrix}$$

Hence, by definition, the equation to the Hessian is $\Delta=0$.

2. Now the equation to the cubic may be written $3xyz+t^3=0\equiv U$, t being the line LMN, and x, y, z the three tangents at L, M, N. Here $A=B=C=0$, $D=6t$, $E=3z$, $F=3y$, $G=3z$, $H=K=L=0$. Hence the equation to the Hessian is

$$H(U)\equiv\begin{vmatrix} 0, & 3z, & 3y, & 0, & a\\ 3z, & 0, & 3x, & 0, & b\\ 3y, & 3x, & 0, & 0, & c\\ 0, & 0, & 0, & 6t, & d\\ a, & b, & c, & d, & 0 \end{vmatrix}=0,$$

which reduces to

$$H(U)\equiv t\left\{a^2x^2+b^2y^2+c^2z^2-2bcyz-2cazx-2abxy\right\}-d^2xyz=0,$$

which is satisfied by $x=0$, $t=0$, and $x=0$, $(by-cz)^2=0$. Hence H(U) touches PQR in the points p, q, r.

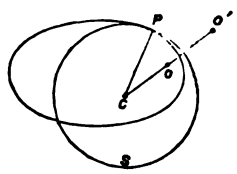
3. Also the equations to the tangents to H(u) at L, M, N are

$d^2x+4bct=0$ (Q′R′), $d^2y+4cat=0$ (R′P′), $d^2z+4abt=0$ (P′Q′).

Hence PP′, QQ′, RR′ meet in the point $ax=by=cz$.

2818. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—Given a circle S and a straight line a not meeting S in real points; O, O′ are the two-point circles to which, and S, a is the radical axis; two conics are drawn osculating S in the same point P, and having one focus at O, O′ respectively: prove that the corresponding directrices coincide.

Solution by MATTHEW COLLINS, B.A.



Let C be the centre of the given circle S; P the origin, the axes of x and y being PC and the tangent at P; then the equation of the given circle whose radius (PC) is r will be

$$x^2+y^2-2rx=0 \quad \ldots\ldots\ldots (1).$$

Now this circle has plainly a three-point osculation at P with the curve

$$y^2-2rx+Ax^2+Bxy+Dx^3+Ey^3+Fx^2y\ldots=0,$$

since both equations give the same value for $\frac{y^2}{x}$ at or rather near P, where x and y are infinitely small, and where therefore x^2, xy, xy^2, y^3, &c. can and should be rejected as infinitely small, compared to the terms retained, viz., those containing x and y^2; we may, therefore, conveniently represent the conic osculating the circle S at P by

$$(a^2-1)x^2+2abxy+(b^2-1)y^2-2rx(b^2-1)=0 \quad \ldots\ldots\ldots (2),$$

which, on account of a and b being indeterminate, can be made to pass through two other given points, or to have a given focus, &c.

Now (2) gives $(ax+by)^2=x^2+y^2+2rx(b^2-1)$,

or, therefore, $(ax+by+c)^2=x^2+y^2+2acx+2bcy+2rx(b^2-1)+c^2$

say $=(x+m)^2+(y+n)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$;

and in this form (3) plainly represents a conic, the coordinates of whose focus are $(-m, -n)$, and the equation of whose directrix is

$$ax+by+c=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Now (3) gives $m^2+n^2=c^2$, $n=bc$, $m=ac+r(b^2-1)$ $\ldots\ldots\ldots\ldots (5)$;

these easily give $b^2=\frac{n^2}{m^2+n^2}$ and $ac=m+r-b^2r=m+r-\frac{rn^2}{m^2+n^2}$,

and thence (4) $\times \frac{c}{n}$ gives

$$\left(\frac{m+r}{n}-\frac{rn}{m^2+n^2}\right)x+y+\frac{m^2+n^2}{n}=0 \quad \ldots\ldots\ldots\ldots (6).$$

Now this equation of the directrix will not change even if m and n (and, therefore, the focus O) do change, *provided that* $\frac{m^2+n^2}{n}$ remains constant, say $=f$ ....(7), and *also that* $\frac{m+r}{n}$ remains constant, say $=g$.....(8).

Now (8) indicates that the point (m, n), that is the focus O, in changing must move upon a straight line passing through the centre C, and (7) indi-

cates that the same point must still lie upon a circle touching PC at P; and hence, if the focus O move *without changing* its corresponding directrix, it can only move to O′, the point *inverse* to O with respect to circle S, COO′ being one straight line, and $CO \cdot CO' = CP^2$.

[A Solution by Mr. TUCKER is given on p. 74 of Vol. XI. of the *Reprint*.]

2873. (Proposed by J. J. WALKER, M.A.)—1. Determine the point D in any quadrant of an ellipse, such that if the osculating circle at it meet the ellipse again in C, that at C will pass through D.

2. Show that A, B, the other two points on the ellipse, the osculating circles at which also pass through D, are the ends of two semi-diameters, each conjugate to that semi-diameter lying in D's quadrant which is equal to the other.

Solution by JAMES DALE.

1. Since the osculating circle at D passes through C, and that at C through D, CD is the common chord of these two circles; and, making equal angles with the tangents at C and D, these tangents must be parallel, and therefore CD must pass through the centre. If (h, k) be the coordinates of D, the equation to the common chord is $y - k = \frac{b^2 h}{a^2 k}(x - h)$, which in this case must be satisfied by $x = 0$, $y = 0$; therefore $a^2k^2 - b^2h^2 = 0$; or, there are four points which fulfil the required condition, viz. the extremities of equal conjugate diameters.

2. Taking $x = \frac{1}{2}a\sqrt{2}$, $y = \frac{1}{2}b\sqrt{2}$ as the coordinates of D, the coordinates of A or B (h, k) are determined by the condition that the line AD or BD makes the same angle with the axis that the tangent at A or B makes; therefore

$$\frac{k - \frac{1}{2}b\sqrt{2}}{h - \frac{1}{2}a\sqrt{2}} = \frac{b^2 h}{a^2 k},$$

whence $(a^2k^2 - b^2h^2)\sqrt{2} = ab(ak - bh)$, $(ak + bh)\sqrt{2} = ab$;

therefore $2a^2k^2 + 4abhk + 2b^2h^2 = a^2b^2 = a^2k^2 + b^2h^2$;

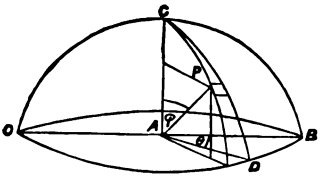
therefore $a^2k^2 + 4abhk + b^2h^2 = 0$;

and if (θ_1, θ_2) be the angles that AD, BD make with the axis, we have from the above equation $\tan\theta_1 \cdot \tan\theta_2 = \frac{b^2}{a^2}$, which proves (2).

2889. (Proposed by J. J. WALKER, M.A.)—The particles of a thin hemispherical shell being supposed to attract with a force varying directly as the distance, prove that the resultant attraction on any point of the rim is the same as if all the matter of the shell were concentrated at the middle point of its axis.

Solution by F. D. THOMSON, M.A.; W. ROBERTS; *the* PROPOSER; *and others.*

Let a be the radius of the shell; then, taking the usual angles at the centre as variables, element of surface at P $= a^2 \sin\phi \, d\phi \, d\theta$, and attraction to O $= \mu \mathrm{OP} a^2 \sin\phi \, d\phi \, d\theta$; therefore attraction of element in direction OA



$= \mu a^3 (1 + \sin\phi \cos\theta) \sin\phi \, d\phi \, d\theta$;

therefore, if X be the total attraction in this direction,

$$X = 2\mu a^3 \int_0 \int_0^{\pi} (\sin\phi + \sin^2\phi \cos\theta) \, d\phi \, d\theta = 2\mu a^3 \pi \int_0^{\frac{1}{2}\pi} \sin\phi \, d\phi = 2\mu\pi a^3.$$

Similarly, if Z be the total attraction parallel to AC,

$$Z = 2\mu a^3 \int_0^{\frac{1}{2}\pi} \int_0^{\pi} \cos\phi \sin\phi \, d\phi \, d\theta = \mu\pi a^3 \int_0^{\frac{1}{2}\pi} \sin 2\phi \, d\phi = \mu\pi a^3;$$

therefore $\frac{Z}{X} = \frac{1}{2}$. Therefore the resultant attraction is towards the middle point of AC, and is equal to $\mu\pi a^3 \sqrt{5}$.

But the distance of this point from O is equal to $\frac{1}{2}a\sqrt{5}$, and whole surface of shell $= 2\pi a^2$; therefore total attraction $= \mu \cdot \frac{1}{2}a\sqrt{5} \times 2\pi a^2$.

[Mr. TUCKER, J. DALE, and others remark that the theorem follows at once from the general property, that "if the particles of a body attract with a force varying as product of the mass into the distance, the resultant attraction is the same as if the whole mass of the body were collected at its centre of gravity" (Todhunter's *Analytical Statics*, p. 246, Cor. 1), which, in this case, is the middle point of the axis.]

2869. (Proposed by J. J. WALKER, M.A.)—1. The square of the chord common to an ellipse and its circle of curvature at any point is equal to $\frac{16}{(a^2-b^2)^2}(a^2-r'^2)(r'^2-b^2)r'^2$, where a, b are the semi-axes, and r' the semi-diameter parallel to the tangent at the point.

2. When the chord is a maximum, $3r'^2 = a^2 + b^2 + (a^4 - a^2b^2 + b^4)^{\frac{1}{2}}$.

3. In the parabola, the square of the common chord is equal to $16p'x$, p' being the parameter, and x the abscissa of the point of contact.

I. *Solution by* JAMES DALE.

1. The equation of a chord common to an ellipse and its osculating circle at the point (h, k) is $b^2h(x-h) - a^2k(y-k) = 0$; and if (h_1, k_1) are the coordinates of the point where the common chord again meets the ellipse, we readily find

$$h - h_1 = \frac{4k^2h}{b^2}, \quad k - k_1 = \frac{4h^2k}{a^2};$$

therefore $$(\text{chord})^2 = 16h^2k^2\left(\frac{h^2}{a^4} + \frac{k^2}{b^2}\right),$$

which, by means of the equation $r'^2 = a^2 + b^2 - (h^2 + k^2)$, becomes

$$(\text{chord})^2 = \frac{16}{(a^2-b^2)^2}(a^2-r'^2)(r'^2-b^2)r'^2.$$

2. Differentiating with respect to r', and putting the result $= 0$, we get $3r'^4 - 2r'^2(a^2+b^2) + a^2b^2 = 0$; therefore $r'^2 = \frac{1}{3}\left\{a^2+b^2 \pm \sqrt{(a^4 - a^2b^2 + b^4)}\right\}$.

The negative sign makes $r'^2 < b^2$, which is impossible; therefore the maximum value of the chord is given by

$$r'^2 = \tfrac{1}{3}\left\{a^2+b^2+\sqrt{(a^2-a^2b^2+b^4)}\right\}.$$

3. The equation to the common chord in the parabola $y^2 = 4ax$ is

$$y - k = -\frac{2a}{12}(x-h), \quad \text{or} \quad y = \frac{2a(3h-x)}{k};$$

and if (h_1, k_1) be the coordinates of the point where the chord again meets the parabola, we have $h_1 = 9h$, and $k_1 = -3k$;

therefore $(\text{chord})^2 = (h-h_1)^2 + (k-k_1)^2 = 64h^2 + 16k^2 = 64h^2 + 64ah$
$= 64a(a+h) = 16p'$. (focal rad. vect. of point of contact).

II. *Solution by the* Proposer.

1. If (x, y) be the point of contact, and (α, β) the other end of the chord in question, it is known that

$$4x^3 - 3a^2x - a^2\alpha = 0, \quad \text{or} \quad x - \alpha = 4\frac{a^2-x^2}{a^2}x = 4\frac{xy^2}{b^2}.$$

Similarly $y - \beta = 4\frac{x^2y}{a^2}$, whence $(x-\alpha)^2 + (y-\beta)^2 = \frac{16}{a^4b^4}(b^4x^2 + a^4y^2)x^2y^2$,

which is the value of the square of the chord in terms of the coordinates of the point of contact.

But $b^4x^2 + a^4y^2 = a^2b^2r'^2$, and $x^2 = \frac{a^2(a^2-r'^2)}{a^2-b^2}$, $y^2 = \frac{b^2(b^2-r'^2)}{b^2-a^2}$.

2. When the chord is a maximum,

$$(a^2-r'^2)(r'^2-b^2) + (a^2-r'^2)r'^2 - (r'^2-b^2)r'^2 = 0,$$

or $$3r'^4 - 2(a^2+b^2)r'^2 + a^2b^2 = 0,$$

whence $$r'^2 = \tfrac{1}{3}\left\{a^2+b^2+(a^4-a^2b^2+b^4)^{\frac{1}{2}}\right\}.$$

[That the positive sign should be taken, appears from this: if the negative were taken, it would follow that

$$3r'^2 = 2(a^2+b^2) + \left\{(a^2-b^2)^2 + a^2b^2\right\}^{\frac{1}{2}}, \quad i.e., \quad 3r'^2 > 3a^2 + b^2.]$$

3. In the case of the parabola $y^2 = px$, it is easily found that

$$\beta = -3y \text{ or } \alpha = 9x, \quad \text{whence} \quad \beta - y = -4y, \quad \alpha - x = 8x,$$

and $$(\alpha-x)^2 + (\beta-y)^2 = 16(y^2+4x^2) = 16(p+4x)x = 16p'x.$$

2920. (Proposed by Professor CAYLEY.) — Imagine a tetrahedron BB′CC′ in which the opposite sides BB′, CC′ are at right angles to each other and to the line joining their middle points M, N; and in which moreover $\overline{CN}^2+\overline{NM}^2+\overline{MB}^2=0$, (or, what is the same thing, the sides CB, CB′, C′B, C′B′ are each $=0$; the tetrahedron is of course imaginary; viz., the lines CC′, BB′ and points M, N may be real; but the distances MB = MB′ and NC = NC′ may be one real and the other imaginary, or both imaginary, but they cannot be both real) the points B, B′ and C, C′ are said to be "skew antipoints." Then it is required to prove that

1. A given system of skew antipoints may be taken to be the nodes (conical points) of a tetranodal cubic surface, passing through the circle at infinity, and which is in fact a Parabolic Cyclide.

2. The equation of the surface may be expressed in the form

$$x(x+\beta)(x+\gamma)+(x+\beta)y^2+(x+\gamma)z^2=0.$$

3. The section through either of the lines ($y=0$, $x+\gamma=0$) and ($z=0$, $x+\beta=0$) is made up of this line and a circle; the two systems of circles being the curves of curvature of the surface; it is required to verify this *à posteriori*; viz., by means of the equation of the surface to transform the differential equation of the curves of curvature in such manner that the transformed equation shall have the integrals

$$y=C(x+\gamma),\quad z=C'(x+\beta).$$

Solution by the Rev. J. WOLSTENHOLME, M.A.

To find the conical points of the surface

$$x(x+\beta)(x+\gamma)+y^2(x+\beta)+z^2(x+\gamma)=0,$$

we have the three equations

$$(x+\beta)(x+\gamma)+x(2x+\beta+\gamma)+y^2+z^2=0,\quad y(x+\beta)=0,\quad z(x+\gamma)=0,$$

which give the four points

$$(x+\beta)=0,\ z=0,\ y^2=\beta(\gamma-\beta);\quad x+\gamma=0,\ y=0,\ z^2=\gamma(\beta-\gamma);$$

and these points lie on the surface. If these points be B, B′, C, C′, and M, N middle points of BB′, CC′, we have

$$BM^2+MN^2+NC^2=\beta(\gamma-\beta)+(\beta-\gamma)^2+\gamma(\beta-\gamma)\equiv 0\quad\text{or}\quad BC=0,$$

and similarly BC′, B′C, B′C′ each $=0$. The points will be two real and two impossible if β, γ have like signs, and will be all impossible if β, γ have unlike signs. The terms of highest dimensions in the equation are $x(x^2+y^2+z^2)$, and therefore the surface contains the circle at infinity.

The equation of the normal at any point (X, Y, Z) of the surface is

$$\frac{-x-X}{(X+\beta)(X+\gamma)+X(2X+\beta+\gamma)+Y^2+Z^2}=\frac{y-Y}{2Y(X+\beta)}=\frac{z-Z}{2Z(X+\gamma)};$$

and where this meets the plane of xy,

$$z=0,\quad y=Y\left(1-\frac{X+\beta}{X+\gamma}\right)=\frac{Y(\gamma-\beta)}{X+\gamma},$$

$$2x=2X-(X+\beta)-X-\frac{X(X+\beta)}{X+\gamma}-\frac{Y^2+Z^2}{X+\gamma}$$

$$=-\beta-\frac{X(X+\beta)}{X+\gamma}-\frac{Y^2(X+\gamma)-Y^2(X+\beta)-X(X+\beta)(X+\gamma)}{(X+\gamma)^2};$$

by equation of the surface, $= -\frac{Y^2(\gamma-\beta)}{(X+\gamma)^2} - \beta$;

therefore $y^2 = (\beta-\gamma)(2x+\beta)$.

So where the normal meets the plane of xz, $y=0$, $z^2 = (\gamma-\beta)(2x+\gamma)$, or the normal meets the planes of xy, xz in two fixed parabolas, either of which is the locus of the vertices of all cones of revolution containing the other. The surface is then a Parabolic Cyclide, and its lines of curvature are circles. This surface is the locus of the points of contact of tangent planes to a series of confocal conicoids drawn from a fixed point in one of the axes. For if

$$\frac{x^2}{k} + \frac{y^2}{k-b^2} + \frac{z^2}{k-c^2} = 1,$$

be one of the conicoids, and tangent planes be drawn through the point $(h, 0, 0)$, we have for the points of contact $\frac{hx}{k} = 1$, and the equation of the locus for different values of k is

$$\frac{x}{h} + \frac{y^2}{hx-b^2} + \frac{z^2}{hx-c^2} = 1,$$

or, removing the origin to the point $(h, 0, 0)$, and altering the notation,

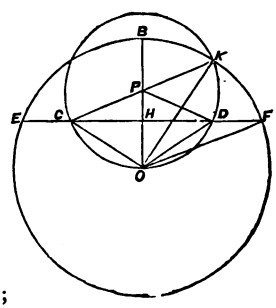
$$x + \frac{y^2}{x-\beta} + \frac{z^2}{x-\gamma} = 0.$$

2296. (Proposed by Professor SYLVESTER.)—If a triangle be formed with the centre of a circle as vertex, and the line joining two points taken at random within the circle as base; show that the chances of the angle at the centre being the greatest, intermediate, or least angle of the triangle, are respectively as

$$4\pi + 6\sqrt{3} : 4\pi - 3\sqrt{3} : 4\pi - 3\sqrt{3}.$$

Solution by STEPHEN WATSON.

Let O be the centre of the circle, OB a radius, P any point in OB, OCPD a rhombus having each side equal to OP. Through C, D draw the chord EF; with P as centre and radius PO describe a circle cutting the given circle in K; and join OK. Put $OB=a$, $OP=x=2OH$. Then in the triangle OPQ the angle O will be the greatest, when Q lies anywhere below EF but without the circle COD, and this area is



$$a^2\left\{\frac{\pi}{2} + \sin^{-1}\frac{x}{2a}\right\} + \tfrac{1}{2}x(a^2 - \tfrac{1}{4}x^2)^{\frac{1}{2}} - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)x^2;$$

hence, multiplying by $2\pi x dx$, integrating between $x=0$, $x=a$, and dividing by $a^4\pi^2$, we get $$p_1 = \frac{1}{3} + \frac{\sqrt{3}}{2\pi} \quad \ldots\ldots (1).$$

When O is the least angle, Q must lie above EF, in the area common to the circles EBF, COG; and when $x > \frac{1}{2}a$, this area is

$$a^2 \cos^{-1}\frac{a}{2x} - x^2\left(\pi - 2\cos^{-1}\frac{a}{2x}\right) - a(x^2 - \tfrac{1}{4}a^2)^{\frac{1}{2}} - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)x^2 \ldots\ldots (2);$$

but when $x < \frac{1}{2}a$, the area is $\left(\frac{2}{3}\pi + \frac{\sqrt{3}}{4}\right)x^2$ ..... (3);

therefore $$p_3 = \frac{1}{a^4\pi^2}\left\{\int_{\frac{1}{2}a}^{a}(1) + \int_{0}^{\frac{1}{2}a}(2)\right\} 2\pi x dx = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} \quad \ldots\ldots (4);$$

and $$p_2 = 1 - p_1 - p_3 = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} \quad \ldots\ldots (5);$$

therefore $$p_1 : p_2 : p_3 = 4\pi + 6\sqrt{3} : 4\pi - 3\sqrt{3} : 4\pi - 3\sqrt{3}.$$

Hints on Gravitation. *By* Septimus Tebay, B.A.

Has matter an innate power of attraction? Of all forces with which we are acquainted, attraction is the only force which is presumed to act on a distant body without a connection. Is this force real, or only apparent? It is now generally admitted that light is produced by the vibrations of an elastic fluid which fills all space. If a small portion of this fluid were annihilated, or the pressure about a point suddenly diminished, there would be a general tendency towards this point from all parts of the surrounding fluid. Let two material particles be placed in this fluid at rest, without weight, or the power of attraction, and acted on by no forces. They will remain at rest for ever. Let us see what would be the consequence if one of them were suddenly set in motion. If the particle be very small, the fluid will not be sensibly disturbed at a considerable distance from the particle; and if the velocity (v) of the particle be great, the velocity of the fluid in the vicinity of the particle will also be great. Adopting the customary notation, we have

$$p = \varpi\epsilon^{-\frac{1}{k}\left(\frac{d\phi}{dt} + \frac{1}{2}U^2\right)}.$$

Now, without inquiring into the nature of the function $\frac{d\phi}{dt}$, which for an elastic fluid is no easy matter, all that is necessary is that $\frac{d\phi}{dt} + \frac{1}{2}U^2$ be positive in the vicinity of the particle. In this case p will be diminished; and if we suppose ρ to be unchanged at a considerable distance from the centre, the motion, or the *tendency* to motion, will be symmetrical about the centre, and will be the same as for an incompressible fluid; that is,

$$r^2 V = f(t).$$

Differentiating with respect to t,

$$\frac{dV}{dt} = \frac{f'(t)}{r^2} - \frac{2V^2}{r}.$$

Now, initially, $V = 0$; hence the instantaneous effect produced is

$$\left(\frac{dV}{dt}\right)_{t=0} = \frac{f'(0)}{r^2};$$

which varies inversely as the square of the distance. If $\frac{d\Phi}{dt}$ were known in any particular case, as for instance a spherical particle, we could determine $f'(0)$; which will be a function of v. If the motion of the particle be uniform, the acceleration will vary inversely as the square of the distance; but if v be variable, the accelerating force will also be variable. In support of these premises some force may be adduced from the fact that comets seem to contract as they approach the sun, that is, as the velocity increases. Here, although the solid nucleus of the comet may have no sensible disturbing effect on a distant body, yet on the flimsy atmosphere of the comet itself this may be otherwise. On this theory we can account for the tail of a comet. Since the sun is in motion, and the effect produced varies inversely as the square of the distance, the density of the fluid will increase as the distance diminishes. Hence, as the comet approaches the sun, its atmosphere, if of less density than that of the surrounding fluid, will recede from the sun, as smoke and vapour rise in the atmosphere of the earth. As the motion of the earth is not quite uniform, it might be expected that this would have some influence on the moon. There is an annual acceleration in the moon's motion of $2''$, which has not been accounted for. May it not be attributed to this cause?

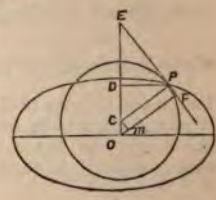
If a solid particle receive a sudden impulse, it will suffer a momentary reaction from the fluid, called inertia; and the equilibrium being restored, and the fluid set in motion, it will continue to flow from fore to aft, and the particle will move on with a uniform motion, without suffering further resistance.

2910. (Proposed by the Rev. J. WOLSTENHOLME, M.A.)—A circle (of radius r) is described with its centre on the minor axis of a given ellipse (of eccentricity e) at a distance from the centre of the ellipse equal to er: prove that the tangents to this circle at any points where it meets the ellipse will touch the minor auxiliary circle.

I. *Solution by* STEPHEN WATSON.

Let O be the centre of the ellipse; C that of the circle; P a point of intersection; EF a tangent to the circle at P meeting OC produced in E, and a perpendicular from O in F; also draw PD perpendicular to OE, and Cm to OF. Put OD $= z$; then $r^2 - (z - er)^2 = \mathrm{DP}^2 = a^2 - \frac{a^2}{b^2}z^2$,

and this quadratic in z gives $cz + \frac{b^2 r}{a} = ab$,



therefore $e(z-er)+r=e.\mathrm{CD}+r=h$;

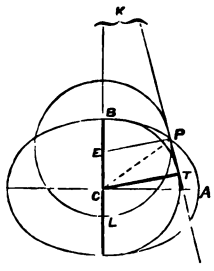
but by similar triangles $\mathrm{CP}:\mathrm{CD}=\mathrm{CO}:Om$, whence $Om=e.\mathrm{CD}$,

therefore $\mathrm{OF}=Om+m\mathrm{F}$, or $\mathrm{CP}=e.\mathrm{CD}+r=h$,

which proves the theorem.

II. *Solution by* R. Tucker, M.A.

Let E be the centre of the variable circle cutting the ellipse in P, and CT a perpendicular from the centre of the ellipse on the tangent to the circle at P; also let $\angle\mathrm{PCT}=\phi$, $\angle\mathrm{ACP}=\theta$.



Then $\dfrac{\sin\phi}{\cos\theta}=\dfrac{\sin\mathrm{CPE}}{\sin\mathrm{PCE}}=\dfrac{\mathrm{CE}}{\mathrm{PE}}=e$;

therefore $\sin\phi=e\cos\theta$ .....(1);

but $\mathrm{CT}=\mathrm{CP}\cos\phi$

$=\dfrac{b}{\sqrt{(1-e^2\cos^2\theta)}}\cdot\sqrt{(1-e^2\cos^2\theta)}=b$;

therefore PT touches the minor auxiliary circle.

2004. (Proposed by W. S. Burnside, M.A.)—Find the condition that the line $\alpha x+\beta y+\gamma z$ may be a normal to the conic

$$(a, b, c, f, g, h)(x, y, z)^2=0.$$

I. *Solution by* W. Spottiswoode, M.A., F.R.S.

The condition that the line $lx+my+nz=0$ may touch the conic is, as is well known, $(\mathrm{A}, \ldots \mathrm{F}, \ldots)(l, m, n)^2=0$,

where, as usual, $\mathrm{A}=bc-f^2$, ... The condition that the line $\alpha x+\ldots=0$ may be perpendicular to $lx+\ldots=0$ is

$$(-1,-1,-1,\cos\mathrm{A},\cos\mathrm{B},\cos\mathrm{C})(\alpha,\beta,\gamma)(ax+hy+gz, hx+by+fz, gx+fy+cz)=0.$$

And, eliminating x, y, z from this equation and the equations to the line and to the conic, and writing for brevity

$$p=\cos\mathrm{A},\quad q=\cos\mathrm{B},\quad r=\cos\mathrm{C},\quad (-1,-1,-1,p,q,r)=(.),$$

$$l=(.)(\alpha,\beta,\gamma)(a,h,g),\quad m=(.)(\alpha,\beta,\gamma)(h,b,f),\quad n=(.)(\alpha,\beta,\gamma)(g,f,c),$$

the condition required becomes

$$(a, \ldots f, \ldots)(\beta n-\gamma m, \gamma l-\alpha n, \alpha m-\beta l)^2=0$$

$$=(cm^2-2fmn+bn^2)\alpha^2+\ldots-2(fl^2+amn-hnl-glm)\beta\gamma-\ldots.$$

Now $cm^2-2fmn+bn^2=\mathrm{A}\{(a\ldots)(-1, r, q)^2\alpha^2+\ldots\ldots$

$\ldots\ldots+2(a\ldots)(r,-1,p)(q,p,-1)\beta\gamma+\ldots\}-\nabla(\alpha+\beta+\gamma)^2$,

whence the total expression may be written as follows:—

$$= (A, \dots F, \dots)(\alpha, \beta, \gamma)^2 . (a, \dots f, \dots)\left\{(-1, r, q)\alpha, (r, -1, p)\beta, (q, p, -1)\gamma\right\}^2 - \nabla(\alpha+\beta+\gamma)^4 = 0.$$

[Mr. BURNSIDE remarks that "the condition may be put under the form

$$\Delta\Omega^2 + \Sigma(\Phi - \Theta_1\Omega) = 0,$$

where $\Omega = \alpha^2+\beta^2+\gamma^2-2\beta\gamma\cos A-2\gamma\alpha\cos B-2\alpha\beta\cos C$, $\Delta\Phi = \Theta\Sigma - \Pi^2$, and the remaining notation as in Quest. 1778," (*Reprint*, Vol. V., p. 27).]

II. *Solution by* R. BALL, M.A.; W. H. LAVERTY, M.A.; *and others.*

Let (x', y', z') be the coordinates of one of the points of intersection of the line with the conic. The tangent to the conic at this point is

$$x(ax'+hy'+gz') + y(hx'+by'+fz') + z(gx'+fy'+cz') = 0;$$

if this be at right angles to $\alpha x + \beta y + \gamma z = 0$,

$$\alpha(ax'+hy'+gz') + \beta(hx'+by'+fz') + \gamma(gx'+fy'+cz') = 0.$$

Write $\quad \alpha^2+\beta^2+\gamma^2 = \rho^2, \quad (a, b, c, d, e, f)(\alpha, \beta, \gamma)^2 = S,$

$$a\alpha+h\beta+g\gamma = L, \quad h\alpha+b\beta+f\gamma = M, \quad g\alpha+f\beta+c\gamma = N;$$

then the problem is reduced to the elimination of x, y, z between the equations

$$Lx+My+Nz = 0, \quad \alpha x+\beta y+\gamma z = 0,$$
$$ax^2+by^2+cz^2+2fyz+2gxz+2hxy = 0.$$

This is the determinant

$$\begin{vmatrix} L, & M, & N, & 0, & 0 \\ \alpha, & \beta, & \gamma, & 0, & 0 \\ a, & h, & g, & \alpha, & L \\ h, & b, & f, & \beta, & M \\ g, & f, & c, & \gamma, & N \end{vmatrix} = 0,$$

which may be reduced to

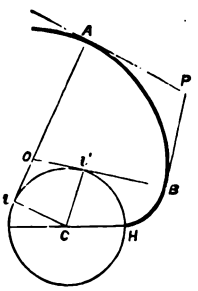
$$\begin{vmatrix} 0, & 0, & 0, & -\rho^2, & -2S \\ \alpha, & \beta, & \gamma, & 0, & -\rho^2 \\ a, & h, & g, & \alpha, & 0 \\ h, & b, & f, & \beta, & 0 \\ g, & f, & c, & \gamma, & 0 \end{vmatrix} = 0, \quad \text{or} \quad 2S\begin{vmatrix} \alpha, & \beta, & \gamma, & 0 \\ a, & h, & g, & \alpha \\ h, & b, & f, & \beta \\ g, & f, & c, & \gamma \end{vmatrix} + \rho^4\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0.$$

2955. (Proposed by M. W. CROFTON, F.R.S.)—1. If T, T′ are tangents drawn from any point to the involute of a circle (radius = 1), and Σ the arc they intercept on the curve, then $T^2 - T'^2 = 2(T + T' - \Sigma)$.

2. Let AOB, A′OB′ be two concentric quadrants, whose sides coincide. If the points A, B′ be joined by the arc AB′ of an involute of a circle, which arc touches the circular arcs at those points, the arc AB′ is an arithmetical mean between the circular arcs AB, A′B′.

I. *Solution by* WILLIAM ROBERTS, JUN.

1. Let $AP=T$, $B'P=T'$, $HCt=\theta$, $HCt'=\theta'$;



then (since radius $=1$) $At=\theta$, $B't'=\theta'$,

$Ot=Ot'=\tan\frac{1}{2}(\theta-\theta')$, $\Sigma=$ arc $AB'=\frac{1}{2}(\theta^2-\theta'^2)$,

$$T^2-T'^2=OB'^2-OA^2=(\theta'+Ot')^2-(\theta-Ot)^2$$
$$=-\theta^2+\theta'^2+2(\theta+\theta')\tan\tfrac{1}{2}(\theta-\theta')$$
$$=2\left\{(\theta+\theta')\tan\tfrac{1}{2}(\theta-\theta')-\Sigma\right\}.$$

But $(\theta+\theta')\tan\frac{1}{2}(\theta-\theta')$
$=(OA+OB')\tan\frac{1}{2}(AOB')=T+T'$,

therefore $T^2-T'^2=2(T+T'-\Sigma)$.

2. Suppose AOB' a right angle, then $\theta-\theta'=\frac{1}{2}\pi$,

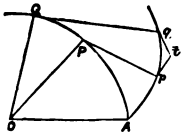
$$OA=\theta-Ot=\theta-1,\quad OB'=\theta'+Ot'=\theta'+1;$$

therefore arc $AB=\frac{1}{2}(\theta-1)\pi$, arc $A'B'=\frac{1}{2}(\theta'+1)\pi$;

therefore arc $AB+$ arc $A'B'=\frac{1}{2}(\theta+\theta')\pi=2\Sigma$.

II. *Solution by the* REV. J. WOLSTENHOLME, M.A.

1. Let the involute start from A; and let p, q be two points in the involute the tangents at which meet in t; O the centre of the generating circle; P, Q points on the circle corresponding to p, q; $AOP=\theta$, $AOQ=\phi$; O origin of rectangular coordinates, axis of x along OA. Then the equations of tangents at p, q are



$$x\sin\theta-y\cos\theta=a\theta,\quad x\sin\phi-y\cos\phi=a\phi;$$

and if (X, Y) be the coordinates of t, we have

$$X\sin(\phi-\theta)=a(\phi\cos\theta-\theta\cos\phi),\quad Y\sin(\phi-\theta)=a(\phi\sin\theta-\theta\sin\phi);$$

whence, if $pt=T$, $qt=T'$,

$$\left.\begin{aligned}T\sin(\phi-\theta)&=a\left\{\phi-\theta\cos(\phi-\theta)-\sin(\phi-\theta)\right\}\\ T'\sin(\phi-\theta)&=a\left\{\theta-\phi\cos(\phi-\theta)+\sin(\phi-\theta)\right\}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(1).$$

Also $2\Sigma=a(\phi^2-\theta^2)$,

$$(T+T')\sin(\phi-\theta)=a(\phi+\theta)\left\{1-\cos(\phi-\theta)\right\},$$
$$(T^2-T'^2)\sin^2(\phi-\theta)=a^2(\phi^2-\theta^2)\sin^2(\phi-\theta)$$
$$-2a^2(\phi+\theta)\sin(\phi-\theta)\left\{1-\cos(\phi-\theta)\right\}$$
$$=2a\Sigma\sin^2(\phi-\theta)-2a(T+T')\sin^2(\phi-\theta),$$

or $$T^2-T'^2=2a(T+T'-\Sigma).$$

2. Take $\phi-\theta=\frac{1}{2}\pi$, then $T'-T=a(2-\frac{1}{2}\pi)$, by equations (1);

therefore $2(T+T'-\Sigma)=(1-\frac{1}{4}\pi)(T+T')$, or $2\Sigma=\frac{1}{4}\pi(T+T')$,

or the arc is an arithmetic mean between quadrants of two circles whose radii are T, T'.

2692. (Proposed by G. M. SMITH, B.A.)—Demonstrate by means of a spherical triangle that the same relation between the amplitudes ϕ, ψ, σ, which gives $F(\phi) + F(\psi) - F(\sigma) = 0$ gives also

$$E(\phi) + E(\psi) - E(\sigma) = \kappa^2 \sin\phi \sin\psi \sin\sigma,$$

when κ is the modulus of the elliptic functions.

Solution by the PROPOSER.

If ϕ, ψ, σ be the sides a, b, c, we have

$$\cos A = \Delta(\phi); \quad \cos B = \Delta(\psi); \quad \cos C = -\Delta(\sigma);$$

$$\kappa = \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Now
$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c};$$

therefore
$$\cos A da + \cos B db + \cos C dc = \frac{(\cos a - \cos b \cos c)\sin a da + \ldots}{\sin a \sin b \sin c}$$

$$= \frac{d(\cos a \cos b \cos c) - \frac{1}{2} d(\cos^2 a + \cos^2 b + \cos^2 c)}{\sin a \sin b \sin c}$$

$$= \frac{d(2\cos a \cos b \cos c - \cos^2 a - \cos^2 b - \cos^2 c)}{2 \sin a \sin b \sin c} \quad \ldots\ldots(1).$$

But
$$\frac{\sin^2 A}{\sin^2 a} = \kappa^2 = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c};$$

therefore (1) may be written

$$\cos A da + \cos B db + \cos C dc = \kappa^2 \frac{d(\sin a \sin b \sin c)^2}{2 \sin a \sin b \sin c} = \kappa^2 d(\sin a \sin b \sin c).$$

Hence, by integrating, and observing that $\int \cos A da = E(\phi)$, &c.,

$$E(\phi) + E(\psi) - E(\sigma) = \kappa^2 \sin\phi \sin\psi \sin\sigma.$$

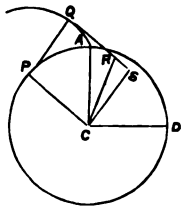
2716. (Proposed by the Rev. JAMES WHITE, M.A.)—If a circular wheel roll against a vertical wall, show that the curve traced on the wheel by a fixed point in the wall is $\theta = \sin^{-1}\frac{d}{\rho} + \frac{(\rho^2 - d^2)^{\frac{1}{2}}}{a}$, where d is the difference between the radius (a) of the circle and the perpendicular height of the fixed point from the ground.

I. *Solution by the* PROPOSER.

The curve may be obtained by considering the circle fixed, and let the ground line roll round it; thus it may be defined as the locus of a point at a fixed distance on the perpendicular to the normal of the involute, as the ground line becomes that normal. (This gives one half the curve on the circle, the other half being equal and opposite.) In other words, it is the locus of the extremity of a tangent of fixed length to the involute.

Taking the particular case in which that length is the radius of the circle, the curve is the Spiral of Archimedes $\rho = a\omega$; for that is the locus of the foot of the perpendicular from centre of circle on tangent to involute, the part intercepted being manifestly equal to the radius. From this the general equation for any length (p) of the tangent can be easily deduced.

Let PQ be the normal and QS the tangent to the involute of the circle PAD.. Then let $QS = CP = a$, and S is a point on the spiral $\rho = a\omega$, and CS is its radius vector. Let $QR = p$, or $RS\,(= a - p) = d$; and it is required to find the equation of the locus of R. CR will be its radius vector.



$$CR^2 = CS^2 + SR^2 = a^2\omega^2 + d^2; \quad i.e., \quad \rho^2 = a^2\omega^2 + d^2;$$

therefore $$\omega = \frac{(\rho^2 - d^2)^{\frac{1}{2}}}{a};$$

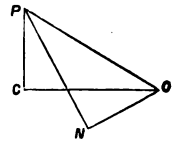
but ω is the angle made by CS with CD (at right angles to CA); therefore, in terms of θ, the angle made by CR (ρ) with CD, $\omega = \theta - \sin^{-1}\frac{d}{\rho}$;

therefore $$\theta = \sin^{-1}\frac{d}{\rho} + \frac{(\rho^2 - d^2)^{\frac{1}{2}}}{a}.$$

By a rack of the form of this curve a moving fulcrum can be controlled. Of this Captain MONCRIEFF has made most valuable and important use in his gun carriage. The sudden and violent strain of the recoil of a heavy gun, before supposed by the best judges to be uncontrollable, has been utilized. The manner in which this curve was practically obtained has caused the rack to be sometimes called cycloidal. But this is a confusion of thought, as the cycloid is generated by a point fixed on the rolling circle, this curve by a point on the wall against which it rolls.

II. *Solution by* C. W. MERRIFIELD, F.R.S.

Let $PC = d$, $PO = r$, $\angle PON = \theta$; also let $\angle CON = \mu$ be the rotation of a diameter, and $CO = a\mu$ the translation of the centre. Then we have at once



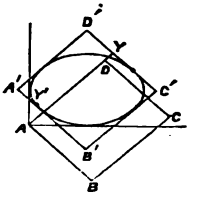
$$r^2 - d^2 = a^2\mu^2,$$

and $$\sin(\theta - \mu) = \sin POC = \frac{d}{r};$$

therefore $$\theta = \sin^{-1}\frac{d}{r} + \frac{(r^2 - d^2)^{\frac{1}{2}}}{a}.$$

2951. (Proposed by M. W. CROFTON, F.R.S.)—A given closed curve, whose equation is $\rho = f(\phi)$, moves without rotation so as always to subtend 90° at a fixed point A. In any position let ABCD be a circumscribed rectangle; show that the envelope of the sides BC or CD will be given by the equation $$\rho = f(\phi + \tfrac{1}{2}\pi) + f(\phi - \tfrac{1}{2}\pi).$$

Solution by R. W. Genese.



Let A'B'C'D' be any circumscribed rectangle; then we get a corresponding position of ABCD by reducing the perpendicular AY ($=p$) on D'C' by DY ($=AY'=p'$), the perpendicular on A'B'. A parallel through D to A'B' gives DC, &c.

Now if ϕ be the inclination of any tangent to the curve to a fixed line through A, p the perpendicular upon it from A, ρ the radius of curvature at the point of contact, we have the relation

$$\rho = p + \frac{d^2p}{d\phi^2}.$$

The equation to the envelope of DC is therefore

$$\rho = (p-p') + \frac{d^2(p-p')}{d\phi^2}.$$

But if, fixing the initial line properly, $p + \frac{d^2p}{d\phi^2} = f(\phi + \frac{1}{2}\pi)$, it is easy to see that

$$p' + \frac{d^2p'}{d\phi^2} = -f(\phi - \tfrac{1}{2}\pi);$$

therefore

$$\rho = f(\phi + \tfrac{1}{2}\pi) + f(\phi - \tfrac{1}{2}\pi).$$

2342. (Proposed by the Rev. R. Townsend, F.R.S.)—Construct the maximum triangle of given species whose three sides shall touch three given circles.

Solution by the Proposer.

Through the centres of the three circles draw three parallels to the three sides of the triangle. The new will evidently be the maximum with the original triangle; but the sides of the new pass through three given points, and the problem is consequently reduced to the one solved in my treatise on *Modern Geometry*, § 55.

2463. (Proposed by C. M. Ingleby, LL.D.)—It is required to divide any rational function of n into two parts, such that the one, and the reciprocal of the other, are similar functions of n and of $n+1$ respectively.

I. *Solution by* W. H. Laverty, M.A.

We have to find $F(n)$ so that

$$F(n) + \frac{1}{F(n+1)} = f(n),$$

$f(n)$ being a known function. Assume $F(0) = k$. Then

$$F(n+1) = \frac{1}{f(n)-F(n)} = \frac{1}{f(n)}\,\frac{-1}{f(n-1)}\,\frac{-1}{f(n-2)} \cdots\cdots \frac{-1}{f(1)}\,\frac{-1}{f(0)-k}$$

$$= \frac{1}{f(n)-P} \text{ suppose;}$$

and the two parts of $f(n)$ are P and $\{f(n)-P\}$.

II. *Solution by the* PROPOSER.

Let p be the given function, and $f(n)$ and $f(n+1)$ the functions involved. Let us put p under the form $q + \frac{1}{q}$; so that we must have

$$q = \tfrac{1}{2}\{p \pm (p^2-4)^{\frac{1}{2}}\}.$$

Then
$$p \equiv \frac{p \pm (p^2-4)^{\frac{1}{2}}}{2} + \frac{2}{p \pm (p^2-4)^{\frac{1}{2}}}.$$

Now, in treating Question 1521, I proved the following theorem, viz.:—

If s be the sum of the odd terms, and s' the sum of the even terms, of a geometrical series whose common ratio is a, and the number of terms be $2n+1$; and σ, σ' be the like when the number of terms are $2(n+1)+1$,

$$\frac{\sigma}{\sigma'} + \frac{s'}{s} = a + \frac{1}{a}.$$

Accordingly, let us put $a = \tfrac{1}{2}\{p \pm (p^2-4)^{\frac{1}{2}}\}$,

then summing both series, we get

$$p = \frac{p \pm (p^2-4)^{\frac{1}{2}}}{2} + \frac{2}{p \pm (p^2-4)^{\frac{1}{2}}}$$

$$= \frac{\left[\tfrac{1}{2}\{p \pm (p^2-4)^{\frac{1}{2}}\}\right]^{2(n+2)}-1}{\tfrac{1}{2}\{p \pm (p^2-4)^{\frac{1}{2}}\}\left\{\left[\tfrac{1}{2}\{p \pm (p^2-4)^{\frac{1}{2}}\}\right]^{n+1}-1\right\}}$$

$$+ \frac{\tfrac{1}{2}\{p \pm (p^2-4)^{\frac{1}{2}}\}\left\{\left[\tfrac{1}{2}\{p \pm (p^2-4)^{\frac{1}{2}}\}\right]^{2n}-1\right\}}{\left[\tfrac{1}{2}\{p \pm (p^2-4)^{\frac{1}{2}}\}\right]^{2(n+1)}-1}$$

or
$$p = f(n+1) + \frac{1}{f(n)}.$$

2844. (Proposed by A. MARTIN.)—To find three positive integral numbers, whose sum, and also the sum of any two of them, shall be a rational cube.

Solution by S. BILLS.

Let $\frac{1}{2}(-x^3+p^3+q^3)$, $\frac{1}{2}(x^3-p^3+q^3)$, $\frac{1}{2}(x^3+p^3-q^3)$ .....(A)
represent the three required numbers; then three of the conditions will be satisfied, and it will only remain to make

$$\frac{1}{2}(x^3+p^3+q^3) = \text{a cube} = r^3, \text{ or } x^3+p^3+q^3 = 2r^3 \quad \ldots\ldots(1).$$

In (1) assume $p = z+1$, $q = z-1$, and $r = z+v$, then we have

$$x^3+2z^3+6z = 2z^3+6z^2v+6zv^2+2v^3,$$

or,
$$z^2+\frac{v^2-1}{v}z = \frac{x^3-2v^3}{6v} \quad \ldots\ldots(2).$$

Solving (2) we find $z = \frac{1-v^2}{2v} \pm \frac{1}{6v}\sqrt{(6vx^3-3v^4-18v^2+9)}$.

We must, therefore, have $6vx^3-3v^4-18v^2+9 = \square$. Assume

$$6vx^3-3v^4-18v^2+9 = 6v(x+2a)(x-a)^2.$$

Developing this and reducing, we find

$$x = \frac{v^4+6v^2+4a^3v-3}{6a^2v}, \text{ and } 6v(x+2a) = \frac{1}{a^2}(v^4+6v^2+16a^3v-3),$$

which must be a square. Assume

$$v^4+6v^2+16a^3v-3 = (v^2+3)^2 = v^4+6v^2+9,$$

whence $v = \frac{3}{4a^3}$; where a may be taken any number that will make *all* the expressions in (A) positive. Assume $a = \frac{1}{2}$, then $v = 6$; and thence $x = 168$, $z = 360$, $p = 361$, $q = 359$; and the three numbers will be 44286264, 1982015, and 2759617. These are the least I have been able to find; others may be found at pleasure.

[After two long but elegant investigations, which we have not room enough to publish, the Proposer gives the two following sets of values, as the least he has been able to find satisfying the conditions of the problem:—

5745716741366	1882652672794926592
39566483258085	743505869778978033
105095301929290	281491575688125711.

The first of these was given on p. 118 of Brickley's *Algebra*, published in Dublin in 1811.]

2928. (Proposed by A. MARTIN.)—Prove that

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n} = n-\frac{n(n-1)}{1.2^2}+\frac{n(n-1)(n-2)}{1.2.3^2}-\text{\&c.}$$

Solution by the Rev. J. BLISSARD.

1. The sum of the reciprocals, viz. $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$, can be equated to an indefinite number of functions of n. (See *Quarterly Journal of Mathematics*, Vol. VI., p. 242.) Thus

$$1+\frac{1}{2}+\ldots+\frac{1}{n}=n-\frac{n(n-1)}{1.2^2}+\frac{n(n-1)(n-2)}{1.2.3^2}-\text{\&c. (as proposed above)}$$

$$=1+\frac{n-1}{1^2.2}-\frac{(n-1)(n-2)}{1.2^2.3}+\frac{(n-1)(n-2)(n-3)}{1.2.3^2.4}-\text{\&c.}$$

$$=\frac{5n-1}{2(n+1)}+\frac{2}{n+1}\left\{\frac{(n-1)(n-2)}{1^2.2^2.3}-\frac{(n-1)(n-2)(n-3)}{1.2^2.3^2.4}+\text{\&c.}\right\}$$

&c. &c. &c.

$$=\frac{1}{n+1}\left\{2n+\frac{1}{1.2}\cdot\frac{n(n-1)}{1.2}-\frac{1}{2.3}\cdot\frac{n(n-1)(n-2)}{1.2.3}+\text{\&c.}\right\}$$

$$=(n-1)+(-1)^n\left\{\frac{n(n-1)}{1.2^2}-2^{n-1}\cdot\frac{n(n-1)(n-2)}{1.2.3^2}+\text{\&c.}\right\}$$

&c. &c. &c.

2. The above Question can also be variously generalized. Thus

$$\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{m+n}=\frac{n}{m+1}-\frac{n(n-1)}{(m+1)(m+2)}\cdot\frac{1}{2}$$

$$+\frac{n(n-1)(n-2)}{(m+1)(m+2)(m+3)}\cdot\frac{1}{3}-\text{\&c.}$$

and

$$=\frac{\Gamma(m+n+1)}{\Gamma(m+1)\Gamma n}\left\{\frac{1}{(m+1)^2}-\frac{n-1}{1}\cdot\frac{1}{(m+2)^2}\right.$$

$$\left.+\frac{(n-1)(n-2)}{1.2}\cdot\frac{1}{(m+3)^2}-\text{\&c.}\right\}.$$

I propose to prove the following generalization, viz. that

$$\frac{x}{1}+\frac{x^2}{2}+\ldots+\frac{x^n}{n}=\frac{n}{1^2}\left\{1-(1-x)\right\}-\frac{n(n-1)}{1.2^2}\left\{1-(1-x)^2\right\}+\text{\&c.}\ldots(\text{I}).$$

Assume

$$\frac{x}{1}+\frac{x^2}{2}+\ldots+\frac{x^n}{n}=\frac{n}{1}\cdot\frac{u_1}{1}-\frac{n(n-1)}{1.2}\cdot\frac{u_2}{2}+\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{u_3}{3}-\text{\&c.}\ \ldots\ (a);$$

therefore $\frac{x}{1}+\frac{x^2}{2}+\ldots+\frac{x^{n+1}}{n+1}=\frac{n+1}{1}\cdot\frac{u_1}{1}-\frac{(n+1)n}{1.2}\cdot\frac{u_2}{2}+\text{\&c.};$

therefore, subtracting and multiplying by $n+1$,

$$\frac{n+1}{1}\cdot u_1-\frac{(n+1)n}{1.2}\cdot u_2+\text{\&c.}=x^{n+1};$$

i. e., putting n for $n+1$, and using Representative Notation,

$$1-(1-u)^n=x^n.$$

Hence, applying Taylor's theorem,

$$f(h+\theta)-f\left\{h+(1-u)\theta\right\}=f(h+x\theta)-fh.$$

Now let $h=1$, $\theta=-1$, and $fz=z^r$;

therefore $-u^r=(1-x)^r-1$, *i. e.*, $u_r=1-(1-x)^r$.

Making this substitution in (a), we obtain the formula (I).

Cor. 1.—In (I), put $1-x$ for x, then

$$\frac{1-x}{1}+\frac{(1-x)^2}{2}+\ldots+\frac{(1-x)^n}{n}=\frac{n}{1^2}(1-x)-\frac{n(n-1)}{1.2^2}(1-x^2)+\&c.$$

Hence, equating coefficients of x^r, we have

$$1+r+\frac{r(r+1)}{1.2}+\ldots+\frac{r(r+1)\ldots(n-1)}{1.2\ldots(n-r)}=\frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}\quad(n>r).$$

Cor. 2.—In (I), put $-x$ for x, then

$$\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\pm\frac{x^n}{n}=\frac{n}{1^2}\left\{(1+x)-1\right\}-\frac{n(n-1)}{1.2^2}\left\{(1+x)^2-1\right\}+\&c.$$

3. We can now readily prove that

$$\frac{x}{1^2}+\frac{x^2}{2^2}+\ldots+\frac{x^n}{n^2}=\frac{n}{1^2}\left\{1-(1-x)\right\}-\frac{n(n-1)}{1.2^2}\left\{1+\frac{1}{2}-(1-x)-\frac{1}{2}(1-x)^2\right\}$$
$$+\frac{n(n-1)(n-2)}{1.2.3^2}\left\{1+\frac{1}{2}+\frac{1}{3}-(1-x)-\frac{1}{2}(1-x)^2-\frac{1}{3}(1-x)^3\right\}-\&c.\ \ldots\ (\mathrm{II}).$$

Let R be the representative of the reciprocals $\frac{1}{1}, \frac{1}{2}, \ldots \frac{1}{n}$, so that $\mathrm{R}^n=\frac{1}{n}$; then, putting $\mathrm{R}x$ for x in (I),

$$\frac{\mathrm{R}x}{1}+\frac{\mathrm{R}^2x^2}{2}+\ldots+\frac{\mathrm{R}^nx^n}{n},\quad i.e.,\quad \frac{x}{1^2}+\frac{x^2}{2^2}+\ldots+\frac{x^n}{n^2}$$
$$=\frac{n}{1^2}\left\{1-(1-\mathrm{R}x)\right\}-\frac{n(n-1)}{1.2^2}\left\{1-(1-\mathrm{R}x)^2\right\}+\ldots\ldots$$
$$\ldots\ldots\pm\frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}\left\{1-(1-\mathrm{R}x)^2\right\}.$$

But $1-(1-\mathrm{R}x)^r=\frac{r}{1}\mathrm{R}x-\frac{r(r-1)}{1.2}\mathrm{R}^2x^2+\&c.=\frac{r}{1^2}x-\frac{r(r-1)}{1.2^2}x^2+\&c.,$

which, from Art. 2, Cor. 1,

$$=\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{r}-\left(\frac{1-x}{1}+\frac{(1-x)^2}{2}+\ldots+\frac{(1-x)^r}{r}\right).$$

Hence, making this substitution, we obtain Formula (II).

Cor. 1.—Put $1-x$ for x in (II), then

$$\frac{1-x}{1^2}+\frac{(1-x)^2}{2^2}+\ldots+\frac{(1-x)^n}{n^2}=\frac{n}{1^2}(1-x)-\frac{n(n-1)}{1.2^2}\left\{1+\frac{1}{2}-\left(x+\frac{x^2}{2}\right)\right\}+\ldots$$
$$\ldots+(-1)^{r+1}\frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}\left\{1+\frac{1}{2}+\ldots+\frac{1}{r}-\left(x+\ldots+\frac{x^r}{r}\right)\right\}+\&c.$$

Hence, equating coefficients of x^r, we have $(n>r)$

$$\frac{1}{r}+\frac{r}{1}\cdot\frac{1}{r+1}+\frac{r(r+1)}{1.2}\cdot\frac{1}{r+2}+\ldots+\frac{r(r+1)\ldots(n-1)}{1.2\ldots(n-r)}\cdot\frac{1}{n}$$
$$=(-1)^{n-r}\left\{\frac{1}{n}-\frac{n}{1}\cdot\frac{1}{n-1}+\frac{n(n-1)}{1.2}\cdot\frac{1}{n-2}-\ldots\pm\frac{n(n-1)\ldots(r+1)}{1.2\ldots(n-r)}\cdot\frac{1}{r}\right\}.$$

Cor. 2.—In (II), put $x = 1$, then

$$\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2} = \frac{n}{1^2}\cdot\Sigma\cdot 1 - \frac{n(n-1)}{1.2^2}\cdot\Sigma\cdot\frac{1}{2} + \frac{n(n-1)(n-2)}{1.2.3^2}\cdot\Sigma\cdot\frac{1}{3}-\ldots$$

where $$\Sigma\frac{1}{n} = 1+\frac{1}{2}+\ldots+\frac{1}{n}.$$

II. *Solution by* W. H. Laverty, B.A.; Rev. J. Wolstenholme, M.A.; *the* Proposer; *and many others.*

We have $$(1-1)^n = 0 = 1-n+\frac{n(n-1)}{1.2}-\ldots\mp n\pm 1,$$

(where the upper signs refer to n as even, and the lower to n as odd,)

therefore $$\frac{1\pm 1}{n} = 1-\frac{(n-1)}{1.2}+\frac{(n-1)(n-2)}{1.2.3}-\ldots\pm 1.$$

Now assume the equality true for $(n-1)$; that is, that

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1} = (n-1)-\frac{(n-1)(n-2)}{1.2^2}+\ldots$$
$$\ldots\pm\frac{(n-1)\ldots(n-k)}{1.2\ldots k^2}\mp\ldots\pm\frac{(n-1)\ldots 1}{1.2\ldots(n-1)^2};$$

also

$$\frac{1+1}{n} = 1-\frac{(n-1)}{1.2}+\ldots\pm\frac{(n-1)\ldots(n-k+1)}{1.2\ldots k}\mp\ldots\pm\frac{(n-1)\ldots 2}{1.2\ldots(n-1)};$$

therefore by addition we obtain

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}+\frac{1\pm 1}{n} = n-\frac{n(n-1)}{1.2^2}+\ldots$$
$$\ldots\pm\frac{n(n-1)\ldots(n-k+1)}{1.2.3\ldots k^2}\mp\ldots\pm\frac{n(n-1)\ldots 2}{1.2\ldots(n-1)^2}$$

or $$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}+\frac{1}{n} = n-\frac{n(n-1)}{1.2^2}+\ldots\mp\frac{n}{n-1}\pm\frac{1}{n}.$$

If, therefore, the equality be true for $(n-1)$, it is true for n; but it is true for 2, 3, &c., therefore it is true generally.

2977. (Proposed by Professor Sylvester.)—Let x be any positive integer, and $1.3.5\ldots(2x-1) = P_x$. Required to prove that

$$xP_{x-1}+\frac{x(x-1)}{2}2P_{x-2}+\frac{x(x-1)(x-2)}{3}2^2P_{x-3}+\ldots+\frac{x(x-1)\ldots 1}{x}2^{x-1}$$
$$= P_x\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2x-1}\right).$$

Solution by J. J. WALKER, M.A.

The left-hand side of the equality to be proved may be written

$$2^{x-1}.1.2.3\ldots x\left(\frac{1.3.5\ldots 2x-3}{1.2.3\ldots x-1}\frac{1}{2_x^{-1}}+\frac{1}{2}\frac{1.3.5\ldots}{1.2.3\ldots}\frac{2x-5}{x-2}\frac{1}{2^{x-2}}\right.$$
$$\left.+\frac{1}{3}\frac{1.3.5\ldots 2x-7}{1.2.3\ldots x-3}\frac{1}{2^{x-3}}+\ldots+\frac{1}{x-1}\frac{1.3}{1.2}\frac{1}{2}+\frac{1}{x}\right)\ldots\ldots\ldots(1).$$

The series within brackets is easily seen to be identical with the coefficient of z in the product of the expansions of $(-1)^{x-1}(1+z)^{-\frac{1}{2}}$ and of $\log_e(1+z)$, which coefficient is otherwise equal to $\dfrac{(-1)^{x-1}}{1.2\ldots x}\dfrac{d^x.y^{-\frac{1}{2}}\log_e y}{dy^x}$, when $y=1$. Now it will readily be found, by induction from the first four or five of the successive differential coefficients of $y^{-\frac{1}{2}}\log_e y$, that

$$\frac{d^x.y^{-\frac{1}{2}}\log_e y}{dy^x}=(-1)^{x-1}\frac{P_x}{2^{x-1}}\,y^{-\frac{2x+1}{2}}\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2x-1}-\frac{\log_e y}{2}\right),$$

which, multiplied by $\dfrac{(-1)^{x-1}}{1.2\ldots x}$ and by the factors outside brackets in (1), becomes identically equal to the right-hand side of the equality to be proved, when $y=1$.

[As examples, give x the values 3 and 4; then

$$3(1.3)+3.2(1)+2.2^2=23=1.3.5+1.5+1.3;$$
$$4(1.3.5)+6.2(1.3)+8.2^2(1)+6.2^3=176=3.5.7+5.7+3.7+3.5.$$

Prof. SYLVESTER remarks that this theorem occurred to him as a corollary to a theorem in his theory of Reducible Cyclodes.]

NOTE ON GRAVITATION. *By* G. O. HANLON.

Some years ago a theory of gravitation occurred to me which has been brought to my mind by Mr. Tebay's paper on Gravitation in the last number of the *Educational Times*. I would first remark, however, on what I consider to be a very weak point in Mr. Tebay's theory, namely, that, under his system, the force of attraction of a body would vary as its volume and not as its mass. The only way in which the mass, independently of the volume, would, in his theory, seem to be able to exercise any effect on other bodies, would be by the power of the body in motion to remove more or less ether from its path in a given time. This, no doubt, depends on its mass; but the hypothesis becomes untenable when we reflect that the sun and planets, all of different densities, have the same velocity round their common centre.

Some philosophers, in order to account for the constant replenishing of the solar heat, maintain that the sun is constantly receiving showers of meteors on its surface. It has been calculated that the quantity of these

bodies which the sun must receive, in order to have his supply of heat maintained, should be sufficient to cover that body all round to the depth of sixty feet in one year. But what is the likelihood of such a supply of heat coming from meteors? Comets, which are of such slight specific gravity, pass very close to the sun (that of 1680 passed within a sixth of his diameter), yet in the memory of man not one has been seen to be absorbed in its mass. I find that, if the meteors which we observe in our own atmosphere are part of a system of meteors moving from all points of the heavens towards the sun, (and it is likely they only occupy that portion of space near the plane of the ecliptic,) that luminary must receive 515 million times the quantity which falls on the earth. Yet we know that very few fall on the earth, while there is every reason to believe that the quantity we meet with every cycle of 33 years is a ring of such meteors moving round, and not into, the sun. This theory does not, therefore, account satisfactorily for the replenishing of the solar heat.

The diminution of the periods of Encke's comet would appear to establish the existence of the *material* ether Mr. Tebay speaks of; and the question naturally arises, would not a motion of the ether towards the sun account for the constant supply of heat, and furnish an explanation of the phenomenon of gravitation. If we suppose every particle of matter in the universe to be constantly absorbing or, as it were, feeding on the surrounding ether, we shall have a motion of that fluid towards the sun and each planet which would account for all the phenomena of gravitation. The force of the ether moving each planet towards the sun would vary inversely as the square of the distance, while towards each of the planets there would be a minor flow of the ether, proportionate to its mass, which would keep the satellites in their orbits. Thus the fundamental requirements of a theory of gravitation would be complied with, the force towards a body varying as the mass, and inversely as the square of the distance, while we are led also towards the solution of the much discussed question of the counteracting influence to the radiation of the sun's heat.

I would further observe that Mr. Tebay draws a conclusion that the nearer to the body the denser the ether. I cannot avoid drawing an opposite conclusion, both in his supposition and mine (except for the part immediately before the moving body). For a long time I endeavoured to prove it would be as he says, since the diurnal motion of the earth, planets, and sun would then be accounted for by the inertia of the ether on the side of a body next the centre of force being greater than that on the opposite side.

2996. (Proposed by J. J. WALKER, M.A.)—If from the three angles of a triangle, which is circumscribed by a conic, lines be drawn parallel to those joining the middle points of the sides opposite those angles respectively with the centre of the conic, the three lines so drawn will cointersect.

Solution by S. WATSON; REV. T. J. SANDERSON, M.A.; F. D. THOMSON, M.A.; R. W. GENESE; *and others.*

This property is true, if, instead of the centre of the conic, we take any point in the plane of the triangle. Let any such point be determined by

$$l\alpha = m\beta = n\gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

then the equations of lines joining this point to the middle points of the sides are $l(cm-bn)\alpha + mn(b\beta - c\gamma) = 0$, $m(an-cl)\beta + nl(c\gamma - a\alpha) = 0$,

$$n(bl-am)\gamma + lm(a\alpha - b\beta) = 0,$$

and if $p = \dfrac{l(bn-cm)}{a}$, $p_1 = \dfrac{m(cl-an)}{b}$, $p_2 = \dfrac{n(am-bl)}{c}$,

the equations of lines through the angles parallel to the preceding are respectively

$$\left.\begin{aligned} (mn+p)\,b\beta &= (mn-p)\,c\gamma \\ (nl+p_1)\,c\gamma &= (nl-p_1)\,a\alpha \\ (lm+p_2)\,a\alpha &= (lm-p_2)\,b\beta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (2).$$

When these meet in a point, we have

$$(mn+p)(nl+p_1)(lm+p_2) = (mn-p)(nl-p_1)(lm-p_2),$$

which reduces to $lmn(lp+mp_1+np_2) + pp_1p_2 = 0$;

and substituting the above values of p, p_1, p_2, the result vanishes identically; hence the lines (2) co-intersect.

[The theorem in the Question is the projection of the theorem that the perpendiculars of a triangle meet in a point, as may be seen by making the circumscribing conic a circle, and then projecting orthogonally the circle into the conic.]

THE "TRUE REMAINDER." *By* ARTEMAS MARTIN.

Let it be required to divide D by d, d being a composite number.

Put $d = d_1d_2d_3d_4 \ldots d_n$; $q_1, q_2, q_3, q_4, \ldots q_n$ = the several quotients respectively; $r_1, r_2, r_3, r_4, \ldots r_n$ = the respective remainders left by the divisors $d_1, d_2, d_3, d_4, \ldots d_n$; and $R_1, R_2, R_3, R_4, \ldots R_n$ = the *true* remainders after each division.

The operation will stand thus:—

$d_1)\,D$

$d_2)\,\underline{q_1} \ldots r_1$, first remainder;

$d_3)\,\underline{q_2} \ldots r_2$, second ,,

$d_4)\,\underline{q_3} \ldots r_3$, third ,,

$q_4 \ldots r_4$, fourth ,,

$\vdots \quad \vdots \quad \vdots$

$d_n)\,\underline{q_{n-1}}$

$q_n \ldots r_n$, last ,,

Reversing the above operation, we have

$$d_1q_1 + r_1 = D, \text{ therefore } R_1 = r_1;$$

$d_2q_2 + r_2 = q_1$, therefore $d_1q_1 = d_1d_2q_2 + d_1r_2$, and $d_1d_2q_2 + d_1r_2 + R_1 = D$.

But $D = d_1d_2q_2 + R_2$, therefore $R_2 = R_1 + d_1r_2 = r_1 + d_1r_2$;

also, $d_3q_3 + r_3 = q_2$, therefore $d_1d_2q_2 = d_1d_2d_3q_3 + d_1d_2r_3$,

and $d_1d_2d_3q_3 + d_1d_2r_3 + R_2 = D$.

But $D = d_1d_2d_3q_3 + R_3$; therefore $R_3 = R_2 + d_1d_2r_3 = r_1 + d_1r_2 + d_1d_2r_3$.
Similarly, we find

$$R_4 = R_3 + d_1d_2d_3r_4 = r_1 + d_1r_2 + d_1d_2r_3 + d_1d_2d_3r_4;$$

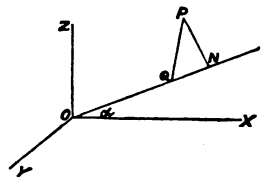
and, *generally*,

$$R_n = R_{n-1} + d_1d_2d_3 \dots d_{n-1}r_n$$
$$= r_1 + d_1r_2 + d_1d_2r_3 + d_1d_2d_3r_4 + \dots + d_1d_2d_3 \dots d_{n-1}r_n,$$

which is the "Rule" given in Arithmetics.

2952. (Proposed by J. J. WALKER, M.A.)—Find the locus of points from which if a heavy inelastic globule be dropped on a smooth inclined plane, its velocity when passing a given horizontal line drawn on the plane shall be constant.

Solution by R. W. GENESE.



If the particle P fall down PQ, striking the inclined plane at Q, the effect of the impulse will be, that the velocity perpendicular to the plane will be destroyed, and that along the plane be unaffected. But if PN be perpendicular to the plane, by the principle of independent motions,

the velocity along plane acquired in falling down PQ = velocity which would be acquired in falling along NQ. Thus the velocity on passing a fixed horizontal line Oy is that which would be acquired in falling from N to O. This is to be constant; therefore N lies on a fixed straight line. Hence the locus of P is a plane perpendicular to given plane.

2909. (Proposed by S. ROBERTS, M.A.)—In a bicircular quartic, the points of contact of the four single tangents drawn from the centre of a circle in which four foci, real or imaginary, lie, are on the circle, and the corresponding points of contact of double tangents also lie on a circle.

Solution by the PROPOSER.

The equation of a quartic with two double points may be written

$$(\alpha\gamma_1 - \alpha_1\gamma)^2(\beta\gamma_1 - \beta_1\gamma)^2 + (m\alpha + n\beta)(\alpha\gamma_1 - \alpha_1\gamma)(\beta\gamma_1 - \beta_1\gamma)\gamma + (a\alpha + b\beta)^2\gamma^2 = 0,$$

the double points being $\widehat{\alpha\gamma}$, $\widehat{\beta\gamma}$. The first polar of $\widehat{\alpha\beta}$ is

$$-2a_1(\alpha\gamma_1 - \alpha_1\gamma)(\beta\gamma_1 - \beta_1\gamma)^2 - 2\beta_1(\beta\gamma_1 - \beta_1\gamma)(\alpha\gamma_1 - \alpha_1\gamma)^2$$
$$+ (m\alpha + n\beta)(-\alpha\beta_1\gamma_1 - \beta a_1\gamma_1 + 2a_1\beta_1\gamma_1)\gamma + 2\gamma(a\alpha + b\beta)^2$$
$$+ (m\alpha + n\beta)(\alpha\gamma_1 - \alpha_1\gamma) = 0.$$

Multiply this by 2γ, and subtract from the equation of the curve. Then

$$2(\alpha\gamma_1-\alpha_1\gamma)(\beta\gamma_1-\beta_1\gamma)\left\{(\alpha\gamma_1-\alpha_1\gamma)(\beta\gamma_1-\beta_1\gamma)+\alpha_1\gamma(\beta\gamma_1-\beta_1\gamma)+\beta_1\gamma(\alpha\gamma_1-\alpha_1\gamma)\right\}$$
$$+(m\alpha+n\beta)(\alpha\beta\gamma_1^2-\alpha_1\beta_1\gamma^2)=0,$$

or
$$\left\{2(\alpha\gamma_1-\alpha_1\gamma)(\beta\gamma_1-\beta_1\gamma)+(m\alpha+n\beta)\gamma\right\}(\alpha\beta\gamma_1^2-\alpha_1\beta_1\gamma^2)=0.$$

But if we change $\alpha_1\beta_1\gamma_1$ into $\alpha_2\beta_2\gamma_2$, the curve represented remaining the same, we have $\alpha_1\beta_1\gamma_2^2-\alpha_2\beta_2\gamma_1^2=0$, that is to say, $\alpha\beta\gamma_1^2-\alpha_1\beta_1\gamma^2=0$ is a conic through $\widehat{\alpha\gamma}$, $\widehat{\beta\gamma}$, on which four intersections of tangents from $\widehat{\alpha\gamma}$, $\widehat{\beta\gamma}$ lie. Hence it follows, &c. The other factor determines the points of contact of the double tangents; these also consequently lie in another conic through $\widehat{\alpha\gamma}$, $\widehat{\beta\gamma}$.

Let now $\widehat{\alpha\gamma}$, $\widehat{\beta\gamma}$ represent the circular points at infinity; γ is the line at infinity, and we have a bicircular quartic; the conics become circles, and $(\alpha_1\beta_1\gamma_1)$ represents a focus.

2749. (Proposed by S. Watson.)—Let D, E, F be the points of contact of the inscribed conic with the triangle of reference, and let AD cut the conic in P_1. Join BP_1, CP_1, meeting the opposite sides in E_1, F_1, and let now D, E_1, F_1 be the points of contact of a second inscribed conic, cutting AD in P_2. Continue this process; then will the equations of the tangent to the nth conic at P_n and of the line $E_n F_n$ be, respectively,

$$l\alpha-\tfrac{1}{2}(4)^n(m\beta+n\gamma)=0, \quad \text{and} \quad l\alpha-(4)^n(m\beta+n\gamma)=0.$$

Solution by the Proposer.

Let the equation of the inscribed conic be

$$(l\alpha)^{\frac{1}{2}}+(m\beta)^{\frac{1}{2}}+(n\gamma)^{\frac{1}{2}}=0 \quad \ldots\ldots\ldots\ldots (1);$$

then the equation of AD is $m\beta-n\gamma=0$, and eliminating first β and then γ from this and (1), the results

$$l\alpha-4n\gamma=0 \text{ and } l\alpha-4m\beta=0 \quad \ldots\ldots\ldots\ldots (2),$$

are the equations of BE_1, CF_1; hence the equation of E_1F_1 is

$$l\alpha-4(m\beta+n\gamma)=0 \quad \ldots\ldots\ldots\ldots (3),$$

and that of the conic touching the sides at D, E_1, F_1 is

$$(l\alpha)^{\frac{1}{2}}+(4m\beta)^{\frac{1}{2}}+(4n\gamma)^{\frac{1}{2}}=0 \quad \ldots\ldots\ldots\ldots (4).$$

Again, the equation of any line through D is $m\beta-n\gamma-r\alpha=0$; and if this meet (1) again in P′, the equation of CP′ is $(l+r)^2\alpha-4lm\beta=0$, hence the equation of P_1P' is

$$l^2\alpha-(2l-r)m\beta-(2l+r)n\gamma=0 \quad \ldots\ldots\ldots\ldots (5),$$

and when $r=0$ this becomes a tangent at P_1, whose equation therefore is

$$l\alpha-2(m\beta+n\gamma)=0 \quad \ldots\ldots\ldots\ldots (6).$$

Proceeding in the same manner with (4) as above with (1), we find the equations of E_2F_2 and the tangent at P_2 to be

$$la-(4)^2(m\beta+n\gamma)=0 \text{ and } la-\tfrac{1}{2}(4)^2(m\beta+n\gamma)=0;$$

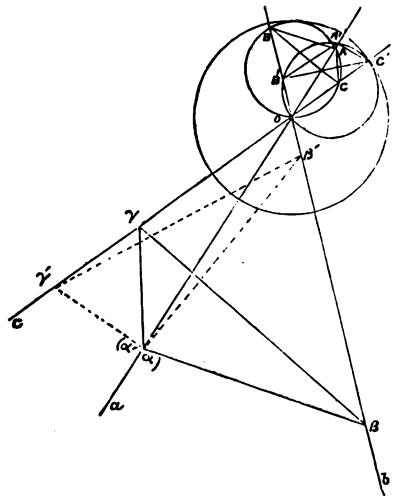
hence, generally, the equations of $E_n F_n$ and the tangent at P_n are

$$la-(4)^n(m\beta+n\gamma)=0 \text{ and } la-\tfrac{1}{2}(4)^n(m\beta-n\gamma)=0.$$

Cor.—The lines $E_1F_1, \ldots E_n F_n$, and the tangents at $P_1, P_2, \ldots P_n$, are all concurrent, as they all pass through the point where the line $m\beta+n\gamma=0$ meets the side BC.

2410. (Proposed by Paul Branquart.)—If a polygon A, B, C, D, ... M be inscribed in a circle, which is supposed to roll on the inside of a circle of double its radius, prove that from the centre o of the greater circle a pencil of m lines $oa, ob, oc, od, \ldots om$ can be drawn such that, for any determined position taken by the given polygon, an infinite number of polygons can be drawn homothetical to it, and having their summits $\alpha, \beta, \gamma, \delta, \ldots \mu$ upon the lines $oa, ob, oc, od, \ldots om$, respectively.

Solution by the Proposer.



Assuming Lahire's theorem (see Dr. Sonnet's *Notions de Mécanique*, p. 10), we remark that, in the revolution of the smaller circle, each summit of the given polygon remains upon a diameter of the greater circle. Consequently, if we consider the pencil formed by the diameters $oa, ob, oc, od, \ldots om$, upon which the summits A, B, C, D, ... M of the given polygon move, it is clear that, *to construct a polygon homothetical to the given one, considered in any determined one of the positions it may assume*, it will be sufficient to take on the diameter oa (considered as an infinite line) any point α, and through α draw $\alpha\beta$ parallel to AB; then through β (β being the intersection of $\alpha\beta$ and ob) draw $\beta\gamma$ parallel to BC; and so on. But the point α can be taken upon oa in an infinite number of ways; hence there are an infinite number of polygons fulfilling the proposed conditions.

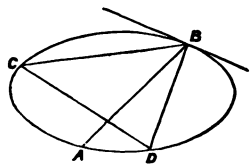
[Mr. Branquart explains his figure thus:—In the above diagram, I have endeavoured to illustrate the construction I have described, and have

considered two different positions of the given polygon (here a triangle, to make the figure plainer), and have with the point α constructed a triangle $\alpha\beta\gamma$ homothetical to ABC, and another $\alpha'\beta'\gamma'$ homothetical to A'B'C'.

Lahire's theorem, referred to above, is thus expressed,—"Si un cercle roule sans glisser sur une circonférence de rayon double et intérieurement à cette circonférence, chacun des points de la petite circonférence se meut sur un diamètre de la grande."]

2803. (Proposed by Professor HIRST.)—Through every point A on a conic pass three circles which osculate the conic elsewhere, say in B, C, D. Prove that A, B, C, D lie on the circumference of a circle, and find the envelope of the latter.

Solution by the Rev. J. WOLSTENHOLME, M.A.



Let BCD be a maximum triangle in an ellipse, and let the circle about BCD meet the ellipse again in A; then BA, CD are equally inclined to axis, therefore BA and tangent at B are equally inclined to axis, therefore circle of curvature at B passes through A; similarly circles of curvature at C, D pass through A. (This proof of this well known proposition was given me by Mr. B. W. Horne, of St. John's, who believed it to have been invented by one of his pupils).

If θ be eccentric angle of A, the equation of the circle ABCD may be found immediately to be

$$2(x^2+y^2) - (a^2-b^2)\left(\frac{x}{a}\cos\theta - \frac{y}{b}\sin\theta\right) = a^2+b^2,$$

and the envelope is therefore the bicircular quartic

$$\left\{2(x^2+y^2)-(a^2+b^2)\right\}^2 = (a^2-b^2)^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right).$$

Hence in any such circle the rectangle of segments of chords drawn through the centre of the ellipse is $\frac{1}{2}(a^2+b^2)$, and the envelope of the polar of the centre of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 4\left(\frac{a^2+b^2}{a^2-b^2}\right)^2.$$

2446. (Proposed by W. K. CLIFFORD, B.A.)—PQ is a chord of a conic, equally inclined to the axis with the tangent at P. Any circle through PQ cuts the conic in RS. Show that the harmonic conjugate of RS relative to P lies on the straight line joining Q to the other extremity of the diameter through P. Hence show by inversion that if chords be drawn to a circular cubic through the point where the asymptote cuts the curve, the locus of their middle points is a circle through the double point.

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The three numbers will be found to be

$$256r^{12}p^8q^8(p^2-q^2)^8(p^2+4pq+q^2)^6\left\{(p^2+q^2)^2\pm 2pq(p^2-q^2)\right\},$$

and $$256r^{12}p^8q^8(p^2-q^2)^8(p^2+4pq+q^2)^6(p^2+q^2)^2.$$

If we take $p=2$, $q=1$, and $r=\frac{1}{2}$, the three numbers will be

$$2^4.3^8.13^6.13, \qquad 2^4.3^8.13^6.25, \qquad 2^4.3^8.13^6.37.$$

Other answers may be found at pleasure.

[If we take $p=2$, $q=-1$, $r=\frac{1}{6}$, we shall obtain the three numbers 1872, 3600, 5328, which will satisfy the conditions of the question, provided we consider one of the square roots *negative*.]

II. *Solution by* Asher B. Evans, M.A.; A. Martin; *and others.*

Let x^2-xy+y^2, x^2+y^2, and x^2+xy+y^2 represent three numbers in arithmetical progression. Assume $x=2mn$, and $y=m^2-n^2$; then it will only remain to make

$$m^2+4mn+n^2=\square \ldots\ldots\ldots\ldots\ldots\ldots(1), \quad \text{and} \quad xy=\text{a cube} \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Assume $(m+pn)$ for the root of (1); then we shall have

$$\frac{m}{n}=\frac{(p+1)(p-1)}{2(2-p)}.$$

Let $p=\frac{3}{2}$, then $m=\frac{5}{4}n$; and condition (2) becomes

$$xy=\tfrac{4}{3}\tfrac{5}{2}n^4=(\tfrac{1}{2}n)^3.2.3^2.5.n=\text{a cube},$$

which condition will be satisfied by taking $n=2^2.3.5^2=300$.

Hence the three numbers are

$$x^2-xy+y^2=1321.3^4.5^8=41797265625,$$
$$x^2+y^2=1681.3^4.5^8=53187890625,$$
$$x^2+xy+y^2=2041.3^4.5^8=64578515625.$$

[These numbers may be obtained from Mr. Bills's expressions by taking $p=5$, $q=4$, $r^{-1}=3.4.11$.]

3003. (Proposed by Professor Sylvester.)—Prove that a uniform circular plate of matter attracting according to the inverse fifth power of the distance will serve to maintain the motion of a particle in the orbit of any circle cutting the circumference of the plate orthogonally, or to speak more strictly, in the portion of such orbit exterior to the plate.

Solution by the Rev. R. Townsend, F.R.S.

This very elegant property, though probably admitting of a more direct demonstration, may be inferred without difficulty from the following considerations:—

1. From Newton's general theorem (*Principia*, Book II., Prop. 7, Cor. 2), combined with a well known property of Dr. Salmon (*Modern Geometry*, Vol. I., Art. 179), it appears that a material point P may be made to describe any circle C, by the action of a force F directed to any fixed

centre O in its plane, and varying directly as the distance of P from O, and inversely as the cube of its distance from the polar L of O with respect to C.

2. By direct integration, or otherwise, it may be readily shown that the attraction F of a uniform circular plate C, whose centre is O and whose mass = M, upon any external point P in its plane, for the law of the inverse 5th power of the distance, $= \frac{M.OP}{PT^6}$, where PT is the length of the tangent from P to the circumference of the plate.

3. For any two fixed circles in the same plane, the square of the tangent PT to either from any point P on the other, varies as the distance PL of P from the radical axis L of the circles (*Modern Geometry*, Vol. I., p. 241); which radical axis L is evidently, when the circles intersect at right angles, the polar of the centre O of either with respect to the other.

From these properties combined—the first of which, it may be observed, was shown by the late Sir W. R. Hamilton to be true, not only for the circle, but for any conic as well—that of Professor Sylvester of course immediately follows.

NOTE.—From the known property, that any orbit, capable of being described under the separate actions of two or more different systems of forces, may be described under the united action of all the systems combined,—the square of the velocity under the conjoint action being equal to the sum of the squares of the velocities under the separate actions at every point of the orbit,—it follows that the above property is true, not only for an entire homogeneous plate, with respect to any circle orthogonal to its circumference, but also for a homogeneous ring bounded by any inner and outer circles, with respect to any circle orthogonal to both; and therefore also for a plate or ring which is not homogeneous all through, but composed instead of homogeneous rings of different densities, bounded by circles which, whether finite or infinite in number, are coaxal, with respect to any circle of the entire system orthogonal to them all.

[Professor SYLVESTER remarks that it would be easy to express the law of density on the hypothesis of the Note as a function of the coordinates in the attracting plate, and interesting to do so.]

2963. (Proposed by W. S. BURNSIDE, M.A.)—At the points where a transversal meets a quartic curve four tangents are drawn, which in general meet the curve again in eight points lying on a conic. Now suppose the quartic curve to be reduced to two conics, and the eight-point conic to become two lines: prove that the envelope of the transversal is a curve of the second class, U = 0, and show that the discriminant of U is $\Delta^2\Delta_1^2(\Theta\Theta_1 - \Delta\Delta_1)^2$ expressed in terms of the invariants of the two conics.

Solution by the Rev. J. WOLSTENHOLME, M.A.

If the transversal $px + qy + rz = 0$ meet the two conics

$$x^2 + y^2 + z^2 = 0 \ldots\ldots(1), \quad lx^2 + my^2 + nz^2 = 0 \ldots\ldots(2),$$

the equation of a conic through the four points where the tangents to (1) at points on the transversal meet (2) is

$$(x^2 + y^2 + z^2)(p^2 + q^2 + r^2) - (px + qy + rz)^2 + k(lx^2 + my^2 + nz^2) = 0 \ldots(A),$$

and the equation of a conic through the four points where the tangents to (2) meet (1) is

$$(lx^2+my^2+nz^2)\left(\frac{p^2}{l}+\frac{q^2}{m}+\frac{r^2}{n}\right)-(px+qy+rz)^2+k'(x^2+y^2+z^2)=0\ldots(\mathrm{B}).$$

(A), (B) obviously coincide if $k=\frac{p^2}{l}+\frac{q^2}{m}+\frac{r^2}{n}$, $k'=p^2+q^2+r^2$, so that the eight points lie on one conic, and the condition that this conic (A) may reduce to two straight lines is

$$(q^2+r^2+kl)(r^2+p^2+km)(p^2+q^2+kn)-2p^2q^2r^2-(q^2+r^2+kl)\,q^2r^2-\ldots\ldots$$
$$\ldots\ldots-\ldots\ldots=0.$$

It is clear that k, k' are factors of the left-hand number, and the other factor is

$$\mathrm{U}\equiv p^2mn(l+m)(l+n)+q^2nl(m+n)(m+l)+r^2lm(n+l)(n+m),$$

or the envelope of the transversal, when the conic through the eight points reduces to two straight lines, is a curve of the second class $\mathrm{U}=0$, whose discriminant is

$$l^2m^2n^2(m+n)^2(n+l)^2(l+m)^2\equiv\Delta^2\Delta_1^2(\Theta\Theta_1-\Delta\Delta_1)^2.$$

2888. (Proposed by S. Roberts, M.A.)—1. If a Cartesian oval has two imaginary axial foci, and consequently two real extra-axial foci, the tangent at any point bisects (internally or externally) the angle between the vector drawn from the real axial focus and the radius of a circle through the extra-axial foci and the point of contact, the radius being drawn thereto.

2. A conic section may be considered as representing a Cartesian oval either with three real axial foci or with two imaginary and one real axial foci. Regarded in the last point of view, what is the property of the conic corresponding to the general case?

Solution by the Proposer.

Take as the equation of the oval, the real axial focus being the pole,

$$\rho^2-2a\rho-2\mathrm{A}x+p^2=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and as the equation of the circle, $\rho^2-2\mathrm{B}x+\mathrm{B}^2-s^2=0\ldots\ldots\ldots\ldots\ldots(2)$.

We must have, as the condition of passing through the extra-axial foci,

$$\frac{\mathrm{A}^2+p^2-a^2}{\mathrm{A}}=\frac{\mathrm{B}^2+p^2-s^2}{\mathrm{B}};$$

hence we get $\rho^2-2a\rho-2\mathrm{A}x+\mathrm{AB}+\frac{\mathrm{B}a^2-\mathrm{A}s^2}{\mathrm{B}-\mathrm{A}}=0.$

Writing ρ', x' for the coordinates of an intersection of (1), (2), we have

$$\rho'=\frac{\mathrm{B}a-\mathrm{A}s}{\mathrm{B}-\mathrm{A}},$$

$$2Bx' = \rho'^2 + B^2 - s^2 = \frac{B^2(a^2-s^2) - 2AB(as-s^2) + B(B-A)^2}{(B-A)^2};$$

therefore
$$\frac{A\rho'}{\rho'-a} = \frac{Ba-As}{a-s},$$

$$\frac{A\rho'}{\rho-a} - x' = \frac{\{2(Ba-As)-(a-s)B\}(B-A)^2-(a-s)^2\{B(a+s)-2As\}}{2(a-s)(B-A)^2}$$
$$= \frac{(B-A)^2-(a-s)^2}{2(a-s)(B-A)^2}\{B(a+s)-2As\};$$

also
$$\frac{A\rho'}{\rho'-a}x' - \rho'^2 = \frac{Ba-As}{a-s}\left\{\frac{\{B(a+s)-2As\}(a-s)+B(B-A)^2}{2(B-A)^2}\right\} - \left(\frac{Ba-As}{B-A}\right)^2$$
$$= \frac{Ba-As}{2(a-s)(B-A)^2}\{(B-A)^2-(a-s)^2\}.$$

But X being the intercept of the tangent on the axis, we have

$$\left\{\frac{A\rho'}{\rho'-a} - x'\right\}X = \frac{A\rho'}{\rho'-a}x' - \rho'^2;$$

whence
$$X = \frac{Ba-As.B}{Ba-As+(B-A)S} = \frac{\rho'B}{\rho'+S}, \text{ or } \frac{B-X}{X} = \frac{s}{\rho'}.$$

But B is the distance of the centre of the circle from the origin, therefore the angle between ρ' and s is bisected.

Suppose now we consider an ellipse as representing a special Cartesian oval with two imaginary axial foci; the real focus is at infinity, on the line of the minor axis. We have then the following theorem:—If a circle be drawn through the real foci of an ellipse, the tangent at a point where the circle meets the ellipse will bisect the angle between the radius of the circle drawn to the point and a line through it parallel to the minor axis. In the case of the hyperbola the theorem also holds, but the angle will be bisected externally.

It is usual to say that a conic is a Cartesian oval of which the third focus goes off to infinity. But it must be remembered that a Cartesian is a bicircular quartic of which one focus has gone off to infinity. So that, in the case of conics, there are two real foci at infinity; and we might define a conic as a Cartesian oval axial in two directions. With reference to a circle, we arrive at the curious conclusion that it represents a Cartesian oval of both kinds with reference to any diameter as axis.

It is plain that both species of Cartesians cut orthogonally when triconfocal; and, indeed, the corresponding theorem for bicircular quartics is also true.

2960. (Proposed by W. K. Clifford, B.A.)—The envelope of a series of surfaces of order n, such that two of them can be drawn through an arbitrary point, is a surface of order $2n$, whose equation may be written in

the form $\surd(\alpha X)+\surd(\beta Y)+\surd(\gamma Z)=0$, where $X=0$, $Y=0$, $Z=0$ are equations of any three surfaces of the series.

The envelope of a net of surfaces of order n, such that two of them can be drawn through *two* arbitrary points, is a surface of order $2n$, whose equation referred to any four surfaces of the net is of the same form as the equation of a quadric referred to four tangent planes.

Solution by R. W. Genese.

The general form of series will be $k^2\phi+k\psi+\chi=0$ .....(1),

where ϕ, ψ, χ are *known* functions of nth degree (for two such surfaces may be found to pass through any given point). Let any three of the series be

$$k_1^2\phi+k_1^2\psi+\chi\equiv X,\quad k_2^2\phi+k_2^2\psi+\chi\equiv Y,\quad k_3^2\phi+k_3^2\psi+\chi\equiv Z;$$

thus (1) may be written $\begin{vmatrix} k^2 & k & 1 & 0 \\ k_1^2 & k_1 & 1 & X \\ k_2^2 & k_2 & 1 & Y \\ k_3^2 & k_3 & 1 & Z \end{vmatrix}=0,$

or $$X(k-k_2)(k-k_3)(k_2-k_3)+\ldots\ldots=0.$$

If we write this $$lX+mY+nZ=0 \quad\ldots\ldots(2),$$

it is easy to see that $$\frac{(k_2-k_3)^2}{l}+\frac{(k_3-k_1)^2}{m}+\frac{(k_1-k_2)^2}{n}=0.$$

This shows, by analogy with the well-known trilinear formulæ, that (2) touches $$(k_2-k_3)\surd X+(k_3-k_1)\surd Y+(k_1-k_2)\surd Z=0.$$

2995. (Proposed by the Rev. J. Wolstenholme, M.A.)—From the top of a tower of height h, particles are projected in all directions in space, with a velocity due to a fall through a height h; show that the mean value of the range is given by the expression

$$2h\int_0^1(1-x^4)^{\frac{1}{2}}\,dx.$$

Solution by the Rev. T. J. Sanderson, M.A.; A. Martin; S. Watson; R. W. Genese; *and others.*

The equation to the trajectory, taking the point of projection as origin and the axis of x parallel to the ground plane, is

$$y=x\tan\alpha-\frac{g}{2V^2\cos^2\alpha}x^2=x\tan\alpha-\frac{x^2}{4h\cos^2\alpha}\quad(\text{since } V^2=2gh).$$

To get the range upon the ground plane, put $y=-h$; then, solving the quadratic, we get $$x=2h\cos\alpha\left\{\sin\alpha\pm(1+\sin^2\alpha)^{\frac{1}{2}}\right\}.$$

These two values of x answer, the one to projection upwards, the other to projection in the opposite direction downwards; and the mean of this pair of values is $2h\cos\alpha\,(1+\sin^2\alpha)^{\frac{1}{2}}$. Hence the mean value of all possible ranges

$$= \frac{2h\Sigma\left\{\cos\alpha\,(1+\sin^2\alpha)^{\frac{1}{2}}\right\}}{n} = \frac{2h\Sigma\left\{\cos\alpha\,(1+\sin^2\alpha)^{\frac{1}{2}}\right\}a^2\cos\alpha\,d\alpha\,d\phi}{na^2\cos\alpha\,d\alpha\,d\phi}$$

(multiplying numerator and denominator by element of surface of sphere)

$$= \frac{2a^2h\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\cos^2\alpha\,(1+\sin^2\alpha)^{\frac{1}{2}}\,d\alpha\,d\phi}{\text{surface of hemisphere}} = \frac{2a^2h.2\pi\int_0^{\frac{1}{2}\pi}\cos^2\alpha\,(1+\sin^2\alpha)^{\frac{1}{2}}\,d\alpha}{2\pi a^2}$$

$$= 2h\int_0^{\frac{1}{2}\pi}\cos^2\alpha\,(1+\sin^2\alpha)^{\frac{1}{2}}\,d\alpha = \text{(putting } x \text{ for } \sin\alpha)\ 2h\int_0^1(1-x^4)^{\frac{1}{2}}\,dx.$$

2767. (Proposed by G. A. Ogilvie.) — If $1+a_1r+a_2r^2+a_3r^3+\ldots$ represents a series of which a_1, a_2, a_3, &c., are the 2nd, 3rd, 4th, &c., terms of the pth order of figurate numbers; show that the sum to n terms of this series is $\dfrac{1-r^n r_p^{-n}}{(1-r)^p}$, where r_p^{-n} denotes the sum of the first p terms in the expansion of r^{-n}.

Solution by the Proposer.

Let $\quad S = 1+a_1r+a_2r^2+a_3r^3+\ldots\ldots+a_{n-1}r^{n-1};$

therefore $\quad rS = r+a_1r^2+a_2r^3+\ldots\ldots+a_{n-2}r^{n-1}+a_{n-1}r^n;$

therefore $\quad (1-r)S = 1+(a_1-1)r+(a_2-a_1)r^2+(a_3-a_2)r^3+\ldots\ldots$
$$\ldots\ldots+(a_{n-1}-a_{n-2})r^{n-1}-a_{n-1}r^n;$$

therefore $\quad (1-r)S = 1+b_1r+b_2r^2+b_3r^3+\ldots\ldots+b_{n-1}r^{n-1}-a_{n-1}r^n\ldots\ldots(\alpha),$

where $a_s-a_{s-1}=b_s=(s+1)$th term in $(p-1)$th order of figurate numbers.

Again, multiplying (α) by r, and subtracting, we have

$$(1-r)^2S = 1+c_1r+c_2r^2+c_3r^3+\ldots+c_{n-1}r^{n-1}-b_{n-1}r^n-a_{n-1}r^n(1-r)\ldots(\beta),$$

where $b_s-b_{s-1}=c_s=(s+1)$th term in $(p-2)$th order of figurate numbers.

Multiplying (β) by r, and subtracting, we have

$$(1-r)^3S = 1+(c_1-1)r+(c_2-c_1)r^2+(c_3-c_2)r^3+\ldots\ldots$$
$$-c_{n-1}r^n-b_{n-1}r^n(1-r)-a_{n-1}r^n(1-r)^2\ldots\ldots(\gamma).$$

Now it is evident from this series that as soon as $1=c_1=c_2=c_3=\ldots=c_{n-1}$,

then all the terms involving c, except the last, will vanish; and this will happen after p operations, since it is of the pth order.

Hence the series becomes

$$(1-r)^p S = 1 - r^n \left\{1 + n(1-r) + \frac{n \cdot (n+1)}{1 \cdot 2}(1-r)^2 + \dots \text{ to } p \text{ terms}\right\} \dots (\delta),$$

for the lowest coefficient of r^n must be 1, and the lowest coefficient but one of r^n must be $n(1-r)$; and, generally, a_n = sum of n terms of pth order

$$= \frac{n \cdot (n-1) \dots (n+p-1)}{1 \cdot 2 \dots r}.$$

The series (δ) can be immediately put in the requisite form

$$(1-r)^p S = 1 - r^n r^{-n}; \quad \text{therefore } S = \frac{1 - r^n r^{-n}_p}{(1-r)^p}.$$

2742. (Proposed by F. D. Thomson, M.A.)—Prove that if tangents be drawn to a cubic from the real points of inflexion, the 12 points of contact lie by *threes* on 19 straight lines.

Solution by the Proposer.

1. Consider the figure formed by two copolar triangles ABC, A′B′C′ such that AA′, BB′, CC′ meet in O, and the corresponding sides of the triangles in L, M, N.

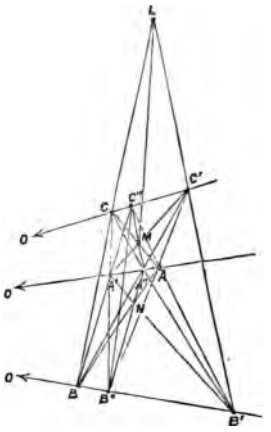
Then it is known that LMN is a straight line.

Now a cubic can be drawn through A, B, C, A′, B′, C′, L, M, N: suppose then that the figure is such that L, M, N are the real points of inflexion of the cubic.

Then since LBC and LNM are straight lines, the satellites of LL, BN, CM are in a straight line.

Therefore L, A, A are in a straight line; or, the tangent at A passes through L.

Similarly, „ A′ „ L,
„ „ B, B′ „ M,
„ „ C, C′ „ N.



Now let A″, B″, C″ be the satellite of AA′, BB′, CC′; then, since the polar conic of a point of inflexion consists of the tangent at the point of inflexion, and another straight line, the tangents at A″B″C″ pass through L, M, N respectively.

Again, since BB′B″ and CC′C″ are straight lines, the satellites of BC, B′C′, B″C″ are in a straight line; therefore L, L, and the satellite of B″C″ are in a straight line; therefore B″C″ passes through L, since L is a point of inflexion. Similarly C″A″ goes through M and A″B″ through N.

2. Again, since AA′A″ and LMN are straight lines, therefore the satellites of AL, A′M, A″N are in a straight line; therefore AC′B″ are in a straight line. Similarly, it can be shown that AC″B′, A′B″C, A′BC″, A″CB′, A″C′B are straight lines.

Hence we have the theorem:—"A chord joining any two of the 12 points of contact of tangents to a cubic which pass through the real points of inflexion passes through a third point of contact." Or, "the 12 points of contact of tangents through the real points of inflexion lie by *threes* on 19 straight lines."

3. It follows from the figure that the pencils $O\{BACL\}$, $O\{ABCM\}$, $O\{ACBN\}$ are harmonic. It is also easily proved analytically from the equation $xyz = (ax+by+cz)^3$, that if PQR be the triangle formed by the tangents at L, M, N, and p, q, r be the points where OA, OB, OC meet the corresponding sides of PQR, then qr passes through L, rp through M, pq through N, and the pencils $P\{QARL\}$ &c. are harmonic.

2942. (Proposed by W. K. Clifford, B.A.)—Let p, q be the foci, and P, Q the asymptotes of a conic; θ the angle it subtends at a point a, and [A] the chord it cuts off from a line A. Then

1. If a line B is drawn through the point a meeting the conic in l, m,

$$al \,.\, am \,.\, \sin \mathrm{BP} \sin \mathrm{BQ} = \frac{ap^2 \,.\, aq^2 \,.\, \sin^2\theta}{pq^2}.$$

2. If from a point b on the line A tangents L, M are drawn to the conic,

$$\sin \mathrm{AL} \sin \mathrm{AM} \,.\, bp \,.\, bq = \frac{\sin^2 \mathrm{AP} \sin^2 \mathrm{AQ} \,.\, [\mathrm{A}]^2}{\sin^2 \mathrm{PQ}}.$$

(Here al means the distance between the points a, l, and BP means the angle between the lines B, P.)

3. Find analogous propositions for a curve of any order on a plane or on a sphere.

Solution (1) *by the* Rev. J. Wolstenholme, M.A.;
(2) *by* J. J. Walker, M.A.

1. If the coordinates of the point a be (X, Y), and the equation of the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the equation, giving the angle $(\tan^{-1} m)$ which a tangent through (X, Y) makes with the axis of x, is

$$(\mathrm{X}^2 - a^2)\, m^2 - 2\mathrm{XY}\, m + \mathrm{Y}^2 - b^2 = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

whence
$$\tan^2\theta = \frac{4\mathrm{X}^2\mathrm{Y}^2 - 4\,(\mathrm{X}^2 - a^2)\,(\mathrm{Y}^2 - b^2)}{(\mathrm{X}^2 - a^2 + \mathrm{Y}^2 - b^2)^2},$$

and
$$\sin^2\theta = \frac{4\,(b^2\mathrm{X}^2 + a^2\mathrm{Y}^2 - a^2b^2)}{(\mathrm{X}^2 + \mathrm{Y}^2)^2 - 2\,(a^2 - b^2)\,(\mathrm{X}^2 - \mathrm{Y}^2) + (a^2 - b^2)^2};$$

also $$ap^2 . aq^2 = \left\{(X-ae)^2 + Y^2\right\}\left\{(X+ae)^2 + Y^2\right\}$$
$$= (X^2 + Y^2 + a^2 - b^2)^2 - 4(a^2 - b^2)X^2$$
$$= (X^2 + Y^2)^2 - 2(a^2 - b^2)(X^2 - Y^2) + (a^2 - b^2)^2,$$

or $$ap^2 . aq^2 . \sin^2\theta = 4(b^2X^2 + a^2Y^2 - a^2b^2) \quad \ldots\ldots\ldots\ldots\ldots (2).$$

Now if ϕ be the angle which the line B makes with the axis of x, its equation will be $$\frac{x-X}{\cos\phi} = \frac{y-Y}{\sin\phi} = r,$$

and for the points where it meets the conic,

$$b^2(X + r\cos\phi)^2 + a^2(Y + r\sin\phi)^2 = a^2b^2,$$

whence $$al . am = \frac{b^2X^2 + a^2Y^2 - a^2b^2}{b^2\cos^2\phi + a^2\sin^2\phi} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3);$$

and, putting i for $\sqrt{(-1)}$, the equations of the asymptotes being

$$y = \pm\frac{b}{a}ix, \text{ we have } \sin^2 BP = \frac{\left(\tan\phi \pm \frac{b}{a}i\right)^2}{(1+\tan^2\phi)\left(1 - \frac{b^2}{a^2}\right)},$$

or $$\sin BP . \sin BQ = \frac{\tan^2\phi + \frac{b^2}{a^2}}{(1+\tan^2\phi)\left(1 - \frac{b^2}{a^2}\right)} = \frac{4(a^2\sin^2\phi + b^2\cos^2\phi)}{pq^2} \ldots (4);$$

or, finally, $$\frac{ap^2 . aq^2 . \sin^2\theta}{pq^2} = al . am . \sin BP \sin BQ.$$

2. This second case may be proved at once by means of the formulæ supplied in the solution of the first case, supplemented by the following. (See Quest. 3025.) Let three lines in the same plane make with any axis which they meet angles $\tan^{-1} m_1$, $\tan^{-1} m_2$, and ϕ, respectively, m_1 and m_2 being given by the quadratic $\alpha m^2 + \beta m + \gamma = 0$; then the product of the sines of the angles which the first two lines make with the third will be given by the formula $$\frac{\alpha\sin^2\phi + \beta\sin\phi\cos\phi + \gamma\cos^2\phi}{\left\{(\alpha-\gamma)^2 + \beta^2\right\}^{\frac{1}{2}}}.$$

From this and equation (1), we have

$$\sin AL . \sin AM = \frac{a^2\sin^2\phi + b^2\cos^2\phi - (X\sin\phi - Y\cos\phi)^2}{\left\{(X^2+Y^2) - 2(a^2-b^2)(X^2-Y^2) + (a^2-b^2)^2\right\}^{\frac{1}{2}}},$$

or $$bp . bq . \sin AL . \sin AM = a^2\sin^2\phi + b^2\cos^2\phi - (X\sin\phi - Y\cos\phi)^2 \ldots (5).$$

Again,

$$[A]^2 = 4\frac{(b^2X\cos\phi + a^2Y\sin\phi)^2 - (b^2X^2 + a^2Y^2 - a^2b^2)(a^2\sin^2\phi + b^2\cos^2\phi)}{(a^2\sin^2\phi + b^2\cos^2\phi)^2}$$
$$= \frac{4a^2b^2}{(a^2\sin^2\phi + b^2\cos^2\phi)}\left\{a^2\sin^2\phi + b^2\cos^2\phi - (X\sin\phi - Y\cos\phi)^2\right\} \ldots (6).$$

Now $$\sin^2 PQ = \frac{4a^2b^2}{(a^2-b^2)^2},$$

and, by (4), $\sin \mathrm{AP} \cdot \sin \mathrm{AQ} = \dfrac{a^2 \sin^2 \phi + b^2 \cos^2 \phi}{a^2 - b^2}$;

whence $\dfrac{4a^2b^2}{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^2} = \dfrac{\sin^2 \mathrm{PQ}}{\sin^2 \mathrm{AP} \cdot \sin^2 \mathrm{AQ}}$;

and, by (6), $a^2 \sin^2 \phi + b^2 \cos^2 \phi - (\mathrm{X} \sin \phi - \mathrm{Y} \cos \phi)^2 = \dfrac{\sin^2 \mathrm{AP} \cdot \sin^2 \mathrm{AQ}\,[\mathrm{A}]^2}{\sin^2 \mathrm{PQ}}$.

Comparing this result with (5), the equality given in Case 2 of the Question is proved.

[3. Mr. CLIFFORD remarks that "the extensions for a plane are

(curve of class n) ..... $al \cdot am \cdot an \ldots \sin \mathrm{BP} \cdot \sin \mathrm{BQ} \ldots$

$$= \frac{(ap \cdot aq \cdot ar \ldots)^{2(n-1)} \sin^2 \mathrm{LM} \cdot \sin^2 \mathrm{LN} \ldots}{pq^2 \cdot pr^2 \cdot qr^2},$$

(curve of order m) ..... $\sin \mathrm{AL} \cdot \sin \mathrm{AM} \ldots bp \cdot bq \ldots$

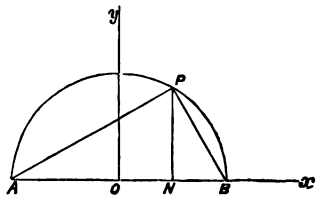
$$= \frac{(\sin \mathrm{AP} \cdot \sin \mathrm{AQ} \ldots)^{2(m-1)} lm^2 \cdot ln^2 \cdot mn^2 \ldots}{\sin^2 \mathrm{PQ} \cdot \sin^2 \mathrm{QR} \cdot \sin^2 \mathrm{QR}}.$$

where P, Q, R ... are the asymptotes, and p, q, r ... the real foci. These give me ideas of the 'distance' of a point from a line or surface, and they may be extended so as to give the distance of two curves from one another."]

2993. (Proposed by A. MARTIN.)—Required the average area of all the right-angled triangles whose hypotenuses are equal to $2a$.

Solution by the REV. T. J. SANDERSON, M.A.; R. W. GENESE; *the* PROPOSER; *and many others.*

The vertices of all the triangles will lie upon a semicircle on the hypotenuse AB.



1. Suppose these vertices distributed in proportion to the arc of this semicircle. Then, if there be n triangles, their average area

$$= \frac{a\Sigma(\mathrm{PN})}{n} = \frac{a\Sigma(\mathrm{PN})\,\delta s}{n\delta s}$$

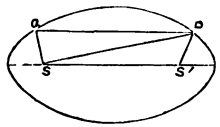
$$= (\text{in limit}) \frac{a}{\pi a} \int_0^{a\pi} y\,\delta s = \frac{1}{\pi} \int_0^{\pi} a^2 \sin\theta\, d\theta = \frac{2a^2}{\pi}.$$

2. Suppose the vertices so distributed that the abscissæ ON &c. increase uniformly. Then the average area of n triangles so described

$$= \frac{a\Sigma(\mathrm{PN})}{n} = \frac{a\Sigma(\mathrm{PN})\,\delta x}{n\delta x} = \frac{a}{a} \int_0^a y\,dx = \frac{\pi a^2}{4}.$$

2964. (Proposed by R. W. Genese.)—The average focal radius-vector of (1) an ellipse, (2) a prolate spheroid, is half the major axis, the distribution being proportional in the former to the arc, in the latter to the surface.

I. *Solution by the* Rev. T. J. Sanderson, M.A.



1. Take P any point in the ellipse, and draw PQ parallel to the major axis.

Now, from the nature of the distribution, it is clear that for every radius-vector SP, drawn to a point P on one side of the minor axis, there corresponds another radius-vector SQ drawn to a corresponding point Q on the other side of the minor axis. And the mean length of this pair of radii-vectores

$$= \tfrac{1}{2}(SP + SQ) = \tfrac{1}{2}(SP + S'P) = \text{half the major axis},$$

and so of every other such pair. Therefore the mean length of *all* the radii-vectores drawn from the focus is equal to half the major axis.

2. For the prolate spheroid, likewise, all the focal radii-vectores may be divided into pairs, such as SP, SQ, whose mean is half the major axis; and therefore the mean of *all* is again equal to half the major axis.

Note.—An exactly similar proof applies to show that if the distribution of the radii-vectores is such that the abscissæ of their extremities increase uniformly, their mean length is still equal to half the major axis.

If the radii are distributed at equal angular intervals, their mean will be found equal to half the *minor* axis. (See Todhunter's *Integral Calculus*, Ch. XIV., Ex. 5.)

II. *Solution by the* Proposer.

1. If x be measured from the centre, we have, in the ellipse,

$$\text{Mean value} = \frac{\int SP.\,ds}{\int ds} = \frac{\int (a+ex)\,ds}{\int ds} = a + e\frac{\int x\,ds}{\int ds};$$

and clearly $\int x\,ds$ over the semi-ellipse $= 0$.

2. In the spheroid, taking as element of surface $2\pi y\,dx$, we have

$$\text{Mean value} = \frac{\int (a+ex)\,2\pi y\,dx}{\int 2\pi y\,dx} = a + e\frac{\int xy\,dx}{\int y\,dx};$$

and we may regard the second term of this expression as $e \times$ (the mean of x) $= 0$. It may be useful to notice that, in one integral, $y\,dx$ is always positive here.

III. *Solution by* Asher B. Evans, M.A.

Let n be the number of positions of P; then, since $SP + S'P = 2a$, we have

$$\frac{\Sigma(SP) + \Sigma(S'P)}{n} = 2a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But it is evident that $\Sigma(SP) = \Sigma(S'P)$ for the entire ellipse; therefore, from (1), $$\frac{2\Sigma(AP)}{n} = 2a;$$ or the mean value of AP is a.

A similar method applies to the spheroid.

2617. (Proposed by R. TUCKER, M.A.)—Solve the differential equation

$$x^2 + y^2 + \frac{2(1+p^2)}{q}(y - xp) = k^2,$$

where $p = \frac{dy}{dx}$, $q = \frac{d^2y}{dx^2}$, and k is a constant.

I. *Solution by the* PROPOSER, *and others.*

Changing to polar coordinates, we get

$$r^3\left(r + \frac{d^2r}{d\theta^2}\right) = k^2\left(r^2 + 2\frac{dx^2}{d\theta^2} - r\frac{d^2r}{d\theta^2}\right);$$

and then writing u for $\frac{1}{r}$, we have

$$\frac{1-k^2u^2}{u(1+k^2u^2)}\frac{d^2u}{d\theta^2} = 1 + \frac{2}{u^2(1+k^2u^2)}\frac{du^2}{d\theta^2},$$

an answer to which is $k^2u^2 + (A\cos\theta + B\sin\theta)u + 1 = 0$,

which is the equation to a circle.

The differential equation arises from the question, "To find the curve such that the tangent to the circle of curvature at every point from a fixed point may be constant."

II. *Solution by* J. J. WALKER, M.A.; W. H. LAVERTY, B.A.; *and others.*

Writing the given equation

$$(x^2+y^2-k^2)q = 2(1+p^2)(xp-y) \quad \ldots\ldots (1),$$

differentiating again, and writing r for $\frac{d^3y}{dx^3}$, there results

$$(x^2+y^2-k^2)r = 6pq(xp-y) \quad \ldots\ldots (2).$$

Dividing (2) by (1), $$\frac{r}{q} = \frac{3pq}{1+p^2}.$$

Multiplying by dx and integrating,

$$\log cq = \tfrac{3}{2}\log(1+p^2), \quad \text{or} \quad cq = (1+p^2)^{\frac{3}{2}} \quad \text{or} \quad \frac{cq}{(1+p^2)^{\frac{3}{2}}} = 1.$$

Multiplying by pdx and again integrating,

$$\frac{c}{(1+p^2)^{\frac{1}{2}}} = y + c', \quad \text{whence} \quad p = \frac{\pm\{c^2-(y+c')^2\}^{\frac{1}{2}}}{y+c'}.$$

Replacing now p by $\frac{dy}{dx}$, inverting, and multiplying by dy,

$$dx = \pm \frac{(y+c')\,dy}{\left\{c^2-(y+c')^2\right\}^{\frac{1}{2}}}.$$

Integrating a third time, $x+c'' = \mp\left\{c^2-(y+c')^2\right\}^{\frac{1}{2}}$.

Squaring and arranging, $x^2+y^2+2c''x+2c'y = c^2$,

writing the single arbitrary c^2 instead of $c^2-c'^2-c''^2$. If, in the given equation, k^2 stands for any arbitrary constant, the above is its complete primitive; otherwise its primitive is

$$x^2+y^2-2c''x+2c'y = k^2,$$

as appears by differentiating twice, eliminating c' and c'', and identifying the result with the proposed equation.

2981. (Proposed by the Rev. G. H. HOPKINS, M.A.)—If the sum of the squares of two consecutive integers be equal to the square of another integer, find their general values, and show how to find any number of particular solutions.

I. *Solution by* MORGAN JENKINS, M.A.

The general solution of $u^2+v^2 = \square$ is $u = m(a^2-b^2)$, $v = m(2ab)$, where a, b are any integers prime to each other, and m is any integer, or, if $a-b$ be even, half any integer.

In this case, therefore, $m = 1$ or $\frac{1}{2}$; whence

$$a^2-b^2-2ab = \pm 1 \text{ or } \pm 2, \quad i.e., \quad (a-b)^2-2b^2 = \pm 1 \text{ or } \pm 2.$$

The general solution of $(a-b)^2-2b^2 = \pm 1$ is, by the application of a theorem of Lagrange's, $\quad a-b = p_x, \quad b = q_x,$

where $\frac{p_x}{q_x}$ is any xth convergent to $\sqrt{2}$ and $p_x^2-2q_x^2 = +1$ and -1 alternately. (We cannot have $a-b = -p_x$, because p_x is $> q_x$.)

This gives $\qquad u = p_x(p_x+2q_x), \quad v = 2q_x(p_x+q_x),$

that is, $\qquad u = p_x p_{x+1}, \qquad v = 2q_x q_{x+1}.$

When the values ± 2 are taken instead of $+1$, we shall only have the values of u and v interchanged. From the differential equation obtained from the law of formation of successive convergents, we have

$$p_x = \tfrac{1}{2}\left\{(1+\sqrt{2})^x+(1-\sqrt{2})^x\right\}, \quad q_x = \frac{1}{2\sqrt{2}}\left\{(1+\sqrt{2})^x-(1-\sqrt{2})^x\right\};$$

and the pairs of integers are (3, 4), (21, 20), (119, 120), &c.

II. *Solution by* ASHER B. EVANS, M.A.; R. W. GENESE; *and others.*

Put $c = a$ and $d = b$ in the identity

$$(a^2+b^2)(c^2+b^2) = (ac-bd)^2+(ab+cd)^2,$$

and it becomes $(a^2+b^2)^2 = (a^2-b^2)^2+(2ab)^2$ .....(1).

The quantities a and b must be so chosen in (1) as to satisfy in integers

$$2ab-(a^2-b^2) = \pm 1 \text{ .....(2).}$$

From (2) we find $b+a = \sqrt{(2a^2 \pm 1)}$ .....(3);

hence $x^2-2a^2 = \pm 1$ .....(4)

must be satisfied in integers. To obtain all the integral values of x and a that will satisfy (4), extract the square root of 2 according to the method of continued fractions, and the terms of each approximate fraction will give one set of values for x and a; thus

$$\frac{x_1}{a_1} = \frac{1}{1}, \quad \frac{x_2}{a_2} = \frac{3}{2}, \quad \frac{x_3}{a_3} = \frac{7}{5}, \quad \frac{x_4}{a_4} = \frac{17}{12}, \quad \text{&c.}$$

But $b = x-a$; therefore $b_1 = 0$, $b_2 = 1$, $b_3 = 2$, &c.

When we have two consecutive values of a and b, other values can readily be found, for

$$\left.\begin{aligned} a_n &= 2a_{n-1}+a_{n-1} \\ b_n &= 2b_{n-1}+b_{n-1} \end{aligned}\right\} \text{ .....(5).}$$

The general values of a and b are

$$a = \mathrm{C}(n,1)+2\mathrm{C}(n,3)+(2^2)\,\mathrm{C}(n,5)+(2^3)\,\mathrm{C}(n,7)+\text{&c. .....(6),}$$

$$b = \mathrm{C}(n-1,1)+2\mathrm{C}(n-1,3)+(2^2)\,\mathrm{C}(n-1,5)+(2^3)\,\mathrm{C}(n-1,7)+\text{&c...(7);}$$

where $$\mathrm{C}(n,r) = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}.$$

By taking n equal to 1, 2, 3, 4, &c., we find, from (1), (6), and (7), all the solutions to the question.

If $n=1$,	$a=1$,	$b=0$,	and $1^2+0^0 = 1^0$;
$n=2$,	$a=2$,	$b=1$,	and $3^2+4^2 = 5^2$;
$n=3$,	$a=5$,	$b=2$,	and $21^2+20^2 = 29^2$;
$n=4$,	$a=12$,	$b=5$,	and $119^2+120^2 = 169^2$;
&c.	&c.		&c.

III. *Solution by the* PROPOSER.

Let x, $x-1$, and y be such integers; then $x^2+(x-1)^2 = y^2$. This equation can be thrown into the form $(2x-1)^2-2y^2 = -1$. By Todhunter's *Algebra*, Art. 639, we have

$$y = \frac{2}{2\sqrt{2}}\left\{(1+\sqrt{2})^n-(1-\sqrt{2})^n\right\}, \quad x = \frac{1}{2}\left[1+\tfrac{1}{2}\left\{(1+\sqrt{2})^n+(1-\sqrt{2})^n\right\}\right]$$

.....(A),

n being any odd integer, and $x=1$, $y=1$ a particular solution. The values of x, y thus obtained are the general values required; other particular solutions, as 4, 5, 21, 29, can be found by putting $n = 3, 5$. The process of expansion becomes very tedious for higher values of n than 5; to obtain,

then, other values of x, y, certain properties of these numbers, easily obtainable from (A), must be employed.

Let x_1, y_1; x_3, y_3; x_5, y_5 ..... $x_{2n'+1}, y_{2n'+1}$ be the values of x, y, when n is put equal to 1, 3, 5 ..... $2n'+1$ in the expressions (A). Then it can be easily proved that

$$y_{2p+1} + y_{2p-1} = 2(x_{2p+1} - x_{2p-1}) \quad \text{.....(B).}$$

But $(x_{2p+1}-1)^2 + x_{2p+1}^2 = y_{2p+1}^2$, and $(x_{2p-1}-1)^2 + x_{2p-1}^2 = y_{2p-1}^2$;

therefore $2(x_{2p+1}-x_{2p-1})(x_{2p+1}-x_{2p-1}-1) = (y_{2p+1}-y_{2p-1})(y_{2p+1}+y_{2p-1})$;

therefore we must have $y_{2p+1} - y_{2p-1} = x_{2p+1} + x_{2p+1} - 1$ .....(C).

From (C) + (B) and (C) − (B) we obtain

$$2y_{2p+1} = 3x_{2p+1} - x_{2p-1} - 1, \quad 2y_{2p-1} = x_{2p+1} - 3x_{2p-1} + 1 \quad \text{..... (1, 2).}$$

From (1), $2y_{2p-1} = 3x_{2p-1} - x_{2p-3} - 1$, $\therefore\ 4y_{2p-1} = x_{2p+1} - x_{2p-3}$ ... (3, 4).

From (2) and (3), by subtraction,

$$x_{2p+1} = 6x_{2p-1} - x_{2p-3} - 2 \quad \text{..... (5).}$$

From (4) and (5),

$$4y_{2p-1} = 6x_{2p-1} - 2x_{2p-3} - 2, \quad x_{2p-1} = 6x_{2p-3} - x_{2p-5} - 2 \quad \text{..... (6, 7).}$$

From (6) and (7), $4y_{2p-1} = 34x_{2p-3} - 6x_{2p-5} - 14$;

therefore $y_{2p-1} = \frac{1}{2}(17x_{2p-3} - 3x_{2p-5} - 7)$, and $x_{2p-1} = 6x_{2p-3} - x_{2p-5} - 2$.

x_{2p-1}	$= 6x_{2p-3} - x_{2p-5} - 2$		y_{2p-1}	$= \frac{1}{2}(17x_{2p-3} - 3x_{2p-5} - 7)$	
x_1	=	1	y_1	=	1
x_3	=	4	y_3	=	5
x_5	= 24 − 1 − 2 =	21	y_5	= ½ (68 − 3 − 7) =	29
x_7	= 126 − 4 − 2 =	120	y_7	= ½ (357 − 12 − 7) =	169
x_9	= 720 − 21 − 2 =	697	y_9	= ½ (2040 − 63 − 7) =	985
x_{11}	= 4182 − 120 − 2 =	4060	y_{11}	= ½ (11849 − 360 − 7) =	5741
x_{13}	= 24360 − 697 − 2 =	23661	y_{13}	= ½ (69020 − 2091 − 7) =	33461

2972. (Proposed by S. Watson.)—OB and OC are any two semi-diameters of an ellipse conjugate to each other; find (1) the locus, and (2) the area, of the curve which is the intersection of normals at B and C.

Solution by R. W. Genese; R. Tucker, M.A.; A. Martin; Rev. J. Wolstenholme, M.A.; *and others.*

1. If θ, $\theta + \frac{1}{2}\pi$ be the eccentric angles of B, C, and $c^2 = a^2 - b^2$, the equations of the normals are

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = c^2, \quad \frac{ax}{\sin\theta} + \frac{by}{\cos\theta} = -c^2;$$

hence, eliminating θ, we readily find the equation of the locus to be

$$2(a^2x^2+b^2y^2)^3 = c^4(a^2x^2-b^2y^2)^2,$$

which is a curve consisting of four loops of equal area.

2. Area $= \iint dx\,dy = \frac{1}{ab}\iint dx'dy'$, where $x' = ax$, $y' = by$,

and therefore $2(x'^2+y'^2)^3 = c^4(x'^2-y'^2)^2$;

or, putting $x' = r\cos\theta$, $y' = r\sin\theta$, we have $r^2 = \frac{1}{2}c^4\cos^2 2\theta$,

$$\text{Area} = \frac{8}{2ab}\int_0^{\frac{1}{4}\pi} r^2 d\theta = \frac{2}{ab}\int_0^{\frac{1}{4}\pi} c^4\cos^2 2\theta\,d\theta = \frac{\pi c^4}{4ab}.$$

Otherwise:

$$\text{Area} = c^4\int_0^{\frac{1}{4}\pi}\frac{(a^2\cos^2\theta - b^2\sin^2\theta)^2\,d\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^3} = c^4\int_0^{\infty}\frac{(a^2-b^2z^2)^2\,dz}{(a^2+b^2z^2)^3}$$

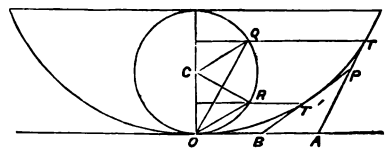
$$= c^4\left\{4a^4\int_0^{\infty}\frac{dz}{(a^2+b^2z^2)^3} - 4a^2\int_0^{\infty}\frac{dz}{(a^2+b^2z^2)^2} + \int_0^{\infty}\frac{dz}{a^2+b^2z^2}\right\}$$

$$= c^4\left\{\frac{3\pi}{4ab} - \frac{\pi}{ab} + \frac{\pi}{2ab}\right\} = \frac{\pi c^4}{4ab}.$$

2998. (Proposed by M. W. Crofton, F.R.S.)—If two tangents to a cycloid include a constant angle, show that their sum has a constant ratio to the included arc of the curve.

I. *Solution by* William Roberts, Jun.

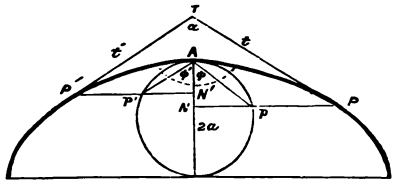
Let PT $= t$, PT′ $= t'$,
arc TT′ $= \Sigma$, $\angle$ QCO $= \theta$,
$\angle$ RCO $= \theta'$;
then $t =$ TA − PA,
$t' =$ PB − BT′;
therefore
$t + t' =$ TA − T′B + PB − PA.



But TA = QO = ½ arc OT = ½s, T′B = RO = ½ arc OT′ = ½s';
therefore
$$\text{TA} - \text{T}'\text{B} = \tfrac{1}{2}\Sigma,$$
also
$$(\text{PB}-\text{PA})\sin\text{BPA} = \text{AB}(\sin\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta')$$
$$= (\text{QT}-\text{RT}')(\sin\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta') = \tfrac{1}{4}(\theta-\theta')(s-s');$$
therefore
$$\text{PB}-\text{PA} = \frac{(\theta-\theta')\,\Sigma}{4\sin\text{BPA}} = \frac{\text{BPA}\,.\,\Sigma}{2\sin\text{BPA}};$$
therefore
$$\frac{t+t'}{\Sigma} = \frac{1}{2}\left(1 + \frac{\text{BPA}}{\sin\text{BPA}}\right),$$
which ratio is constant if the angle BPA is constant.

II. *Solution by* R. W. Genese.

With the notation of the fi- we have



$t\cos\phi - t'\cos\phi' = NN'$

$= 2a\cos^2\phi - 2a\cos^2\phi'$

$= 2a\sin(\phi'-\phi)\sin(\phi'+\phi)$,

$t\sin\phi + t'\sin\phi' = PN + P'N'$

$= 2a\sin(\phi'+\phi)\cos(\phi'-\phi) + 2a(\pi-\phi-\phi')$;

therefore $\frac{t}{2a}\sin(\phi'+\phi) = \sin(\phi'+\phi)\cos\phi + (\pi-\phi-\phi')\cos\phi'$;

and $\frac{t'}{2a}\sin(\phi'+\phi) = \sin(\phi'+\phi)\cos\phi' + (\pi-\phi-\phi')\cos\phi$;

therefore, putting $\phi+\phi' =$ angle between t and $t' = \alpha$,

we have $\frac{t+t'}{2a}\sin\alpha = (\sin\alpha + \pi - \alpha)(\cos\phi + \cos\phi')$;

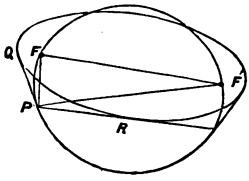
but arc $AP = 2Ap = 4a\cos\phi$, and $AP' = 4a\cos\phi'$,

therefore $\frac{t+t'}{\text{arc } PP'} = \frac{\sin\alpha + \pi - \alpha}{2\sin\alpha} =$ a constant.

2744. (Proposed by the Rev. J. Wolstenholme, M.A.)—A circle passes through the real foci of a conic, prove that the points where the common tangents touch the circle lie on one of the tangents to the conic at the ends of the minor axis.

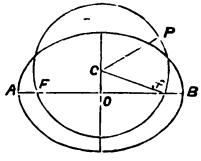
I. *Solution by* W. S. McCay, B.A.

Draw the tangent PR; then $\angle QPF = F'PR$, and $\angle QPF = FF'P$, since QP touches the circle, therefore $\angle F'PR = FF'P$; and therefore PR is parallel to FF'. Hence the theorem follows.



II. *Solution by* Stephen Watson.

Let O, C be the centres of an ellipse and a circle passing through its foci F, f, and P the point of contact of a common tangent with the circle. Put $OC = m$, $\angle PCO = \frac{1}{2}\pi + \phi$, and take a, b, c as usual. Then the coordinates of P are



$x_1 = (c^2+m^2)^{\frac{1}{2}}\cos\phi$, and $y_1 = m + (c^2+m^2)^{\frac{1}{2}}\sin\phi$ .....(1);

also the equation of a tangent to the ellipse perpendicular to CP is

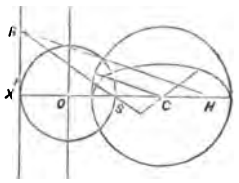
$$x + y\tan\phi = (a^2 + b^2\tan^2\phi)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The condition that this shall pass through (1) gives a quadratic in $\sin\phi$, from which $(c^2 + m^2)^{\frac{1}{2}}\sin\phi + m = (a^2 - c^2)^{\frac{1}{2}} = b$,

that is, $y_1 = b$, which proves the proposition, as the proof is similar in the hyperbola.

2880. (Proposed by R. TUCKER, M.A.)—Given the focus, directrix, and length between the extremities of the major and minor axes of an ellipse; show how to find the major axis and the directions of conjugate diameters by an easy geometrical construction.

Solution by the PROPOSER.



Let S be the nearer focus, and O the foot of the perpendicular from S on the directrix. Draw a circle with centre O and radius OS. Next describe an orthogonal circle to this, having its centre in OS produced, and radius = the given distance between the extremities of the axes; and let its centre be C: this is the centre of the ellipse. H, the other focus, can now be found; and the tangent to the original circle from H is equal to the major axis. Again, if any point R be taken on a tangent to the first circle where SO cuts it (at X′ suppose) and joined to S and H; then lines through C, parallel to HR and perpendicular to RS respectively, give the directions of conjugate diameters.

2786. (Proposed by J. J. WALKER, M.A.)—If ρ be the angular radius of a primary or secondary rainbow, corresponding to angles (θ, θ') of incidence and reflexion at the surface of the rain-drop, and n be the number of internal reflexions, prove that

$$\cos\tfrac{1}{2}(n\pi - \rho) = \cos(\tfrac{1}{2}n\pi - \theta)(n\sin\theta')^{n+1}.$$

Solution by the PROPOSER.

In the case of the primary bow, it is well known that $\frac{1}{2}\rho = 2\theta' - \theta$, where θ, θ' are determined by the equations

$$\sin\theta = \mu\sin\theta', \quad 2\cos\theta = \mu\cos\theta' \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Hence $\sin\frac{1}{2}\rho = \sin(2\theta' - \theta) = (2\sin\theta'\cos\theta - \sin\theta\cos\theta')\cos\theta' + \sin\theta\sin^2\theta'$.

The terms within brackets vanish, in virtue of equation (1); hence

$$\sin\tfrac{1}{2}\rho = \cos\tfrac{1}{2}(\pi - \rho) = \sin\theta\sin^2\theta' = \cos(\tfrac{1}{2}\pi - \theta)(\sin\theta')^2.$$

Again, in the case of the secondary bow, $\frac{1}{2}\pi - \frac{1}{2}\rho = 3\theta' - \theta$, where

$$\sin\theta = \mu \sin\theta', \quad 3\cos\theta = \mu\cos\theta' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Hence $\cos\frac{1}{2}\rho = \sin(3\theta' - \theta)$

$$= 3\sin\theta'\cos\theta - \sin\theta\cos\theta' + 4\sin^2\theta'(\sin\theta\cos\theta' - \sin\theta'\cos\theta).$$

The first two terms vanish, and those within brackets are equal to $2\sin\theta'\cos\theta$, in virtue of equations (2).

Hence $\cos\frac{1}{2}\rho = 8\sin^3\theta'\cos\theta$, or $\cos\frac{1}{2}(2\pi - \rho) = \cos(\frac{1}{2}\pi - \theta)(2\sin\theta')^3$.

The formula does not hold for the tertiary bow.

2668. (Proposed by W. S. Burnside, M.A.)—If the complete solution of the differential equation $\frac{d^n y}{dx^n} + f(x)\frac{d^{n-1}y}{dx^{n-1}} + \mathrm{F}\left(\frac{d^{n-2}y}{dx^{n-2}} \ldots \frac{dy}{dx}, y, x\right) = 0$ be $y = \phi(x, x_1, x_2, \ldots x_n)$, where x_1, x_2, &c. are the n arbitrary constants introduced by integration; prove that

$$\begin{vmatrix} \frac{d\phi}{dx_1} & \frac{d\phi}{dx_2} & \cdots & \frac{d\phi}{dx_n} \\ \vdots & \vdots & & \vdots \\ \frac{d\phi^{(n)}}{dx_1} & \frac{d\phi^{(n)}}{dx_2} & \cdots & \frac{d\phi^{(n)}}{dx_n} \end{vmatrix} \epsilon^{\int f(x)\,dx} = \text{constant},$$

where $\phi^{(r)} = \frac{d^r\phi}{dx^r}$.

Solution by the Proposer.

First, it is well known that

$$\begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} & \cdots & y_n^{(2)} \\ \vdots & \vdots & \vdots & & \vdots \\ y_1^{(n)} & y_2^{(n)} & y_3^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} \epsilon^{\int f(x)\,dx} = \text{constant},$$

where $y_1, y_2, y_3, \ldots y_n$ are the n particular integrals of the linear differential equation $\frac{d^n y}{dx^n} + f(x)\frac{d^{n-1}y}{dx^{n-1}} + f_1(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots + f_{n-1}(x)y = 0$ .....(1).

Secondly, if $y = \phi(x, x_1, x_1 \ldots x_n)$ be the complete solution of the differential equation $\Gamma(x, y, y', y'' \ldots y^{(n)}) = 0$, the complete solution of the linear differential equation

$$\frac{d^n z}{dx^n}\cdot\frac{d\Gamma}{dy^{(n)}}+\frac{d^{n-1}z}{dx^{n-1}}\cdot\frac{d\Gamma}{dy^{n-1}}+\frac{d^{n-2}z}{dx^{n-2}}\cdot\frac{d\Gamma}{dy^{n-2}}+\ldots\ldots+z\cdot\frac{d\Gamma}{dy}=0\ldots(2)$$

is $$z=c_1\frac{d\phi}{dx_1}+c_2\frac{d\phi}{dx_2}+\ldots\ldots+c_n\frac{d\phi}{dx_n}.$$

In the present case, where $\Gamma(x, y, y'\ldots y^{(n)})\equiv y^n+f(x)\,y^{(n-1)}+\ldots$, the linear differential equation (2) becomes

$$\frac{d^n z}{dx^n}+f(x)\frac{d^{n-1}z}{dx^{n-1}}+\ldots\ldots+\psi(x)\,.\,z=0.$$

Now, applying the first theorem to this equation, remembering that $\frac{d\phi}{dx_1}, \frac{d\phi}{dx_2}, \ldots\ldots, \frac{d\phi}{dx_n}$ take the place of $z_1, z_2, z_3, \ldots z_n$, we have the theorem in the question proved.

2734. (Proposed by J. Wilson.)—Prove that

$$n^{n+1}=(n+1)\,S_n-\frac{(n+1)\,n}{1.2}\,S_{n-1}+\frac{(n+1)\,n\,(n-1)}{1.2.3}\,S_{n-2}-\ldots\pm S_0,$$

where $S_1, S_2, S_3, \ldots S_n$ are the sums of the first, second, third, &c. powers of the terms of the series 1, 2, 3, 4, ... to n terms.

Solution by the Proposer.

We have $$S_{n+1}-\frac{n+1}{1}S_n+\frac{(n+1)\,n}{1.2}S_{n-1}-\ldots\pm S_0$$

$$=(1^{n+1}+2^{n+1}+\ldots n^{n+1})-\frac{n+1}{1}(1^n+2^n+3^n+\ldots n^n)$$

$$+\frac{(n+1)\,n}{1.2}(1^{n-1}+2^{n-1}+\ldots n^{n-1})-\ldots\pm(1+1+1\ldots\text{to } n \text{ terms})$$

$$=(1-1)^{n+1}+(2-1)^{n+1}+\ldots(n-1)^{n+1}=1^{n+1}+2^{n+1}+\ldots(n-1)^{n+1}.$$

Hence we find

$$S_{n+1}-\left\{1^{n+1}+2^{n+1}+\ldots(n-1)^{n+1}\right\}=n^{n+1}$$

$$=\frac{n+1}{1}S_n-\frac{(n+1)\,n}{1.2}S_{n-1}+\ldots\pm S_0=(n).$$

2938. (Proposed by T. Cotterill, M.A.)—Let X, Y, Z be the intersections of the three pairs of lines through four points A, B, C, D. Then a line in their plane contains a pair of points harmonic conjugates to the

...ents of su... ...
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conics through ABCD. Show that the locus of such points on the tangents to a curve of the kth class is a curve of the $(3k)$th order, having multiple points of the order k at each of the seven points A, B, C, D, X, Y, Z, the tangents at each of these points being easily found. If the first curve touch a line through two of the points A, B, C, D, the second curve contains this line. If the first curve is a conic through the same four points, the second curve contains the conic.

Solution by the Rev. J. WOLSTENHOLME, M.A.

If the four points A, B, C, D be taken to be $x^2 = y^2 = z^2$, the equation of the series of conics will be $lx^2 + my^2 + nz^2 = 0$, with the condition $l + m + n = 0$; and the two points $(x_1y_1z_1)$, $(x_2y_2z_2)$ will be conjugate to this series of conics, if $x_1x_2 = y_1y_2 = z_1z_2$. Hence on any straight line $px + qy + rz = 0$ the two points given by the equations $px + qy + rz = 0$, $\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 0$ will be conjugates. If now this straight line be a tangent to a given curve of the kth class, p, q, r are connected by an equation $f(p, q, r) = 0$ of the kth degree; and eliminating, we have for the locus of the points on all such tangents

$$f\{x(y^2-z^2),\ y(z^2-x^2),\ z(x^2-y^2)\} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

a curve of the $3k$th order for their locus.

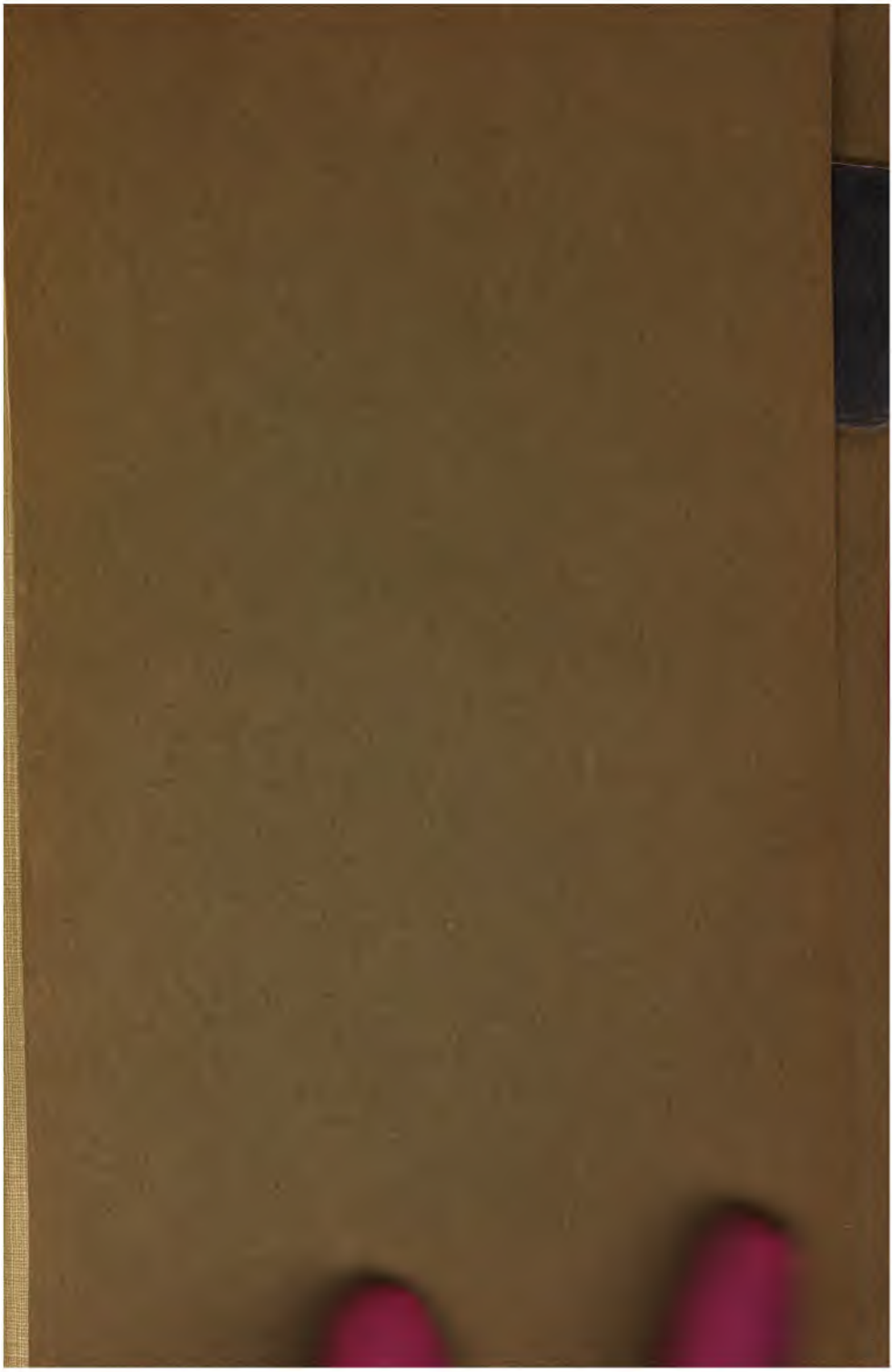
At the four points A, B, C, D the equations $px + qy + rz = 0$, $pyz + qzx + rxy = 0$ coincide; and these points are therefore points on the locus for each of the k tangents drawn through the point, or each is a multiple point of the order k. At the point (X, Y, Z) the equation $pyz + qzx + rxy = 0$ is always satisfied; and these are then also multiple points of the order k. The first curve will touch a line through two of the four points A, B, C, D if $f(0, 1, -1) = 0$, and in this case the line $(y = z)$ through the two points satisfies the equation (A). If the first curve be a conic through the four points, the relation between p, q, r is $\frac{p^2}{l} + \frac{q^2}{m} + \frac{r^2}{n} = 0$, with the condition $l + m + n = 0$. In this case the locus is

$$\frac{x^2(y^2-z^2)^2}{l} + \frac{y^2(z^2-x^2)^2}{m} + \frac{z^2(x^2-y^2)^2}{n} = 0;$$

and when this meets $lx^2 + my^2 + nz^2 = 0$, we have

$$-(my^2+nz^2)(y^2-z^2)^2 + \frac{y^2}{m}(lz^2+my^2+nz^2)^2 + \frac{z^2}{n}(ly^2+my^2+nz^2)^2 = 0$$

or
$$-(my^2+nz^2)(y^2-z^2)^2 + \frac{y^2}{m}(my^2-mz^2)^2 + \frac{z^2}{n}(nz^2-ny^2)^2 = 0,$$

by reason of the condition $l + m + n = 0$. But this equation is identically true, or the locus contains the conic.

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